Math 417: Midterm 2
Tuesday, November 5, 2019

Instructions:

• Write your name at the top of this page.

• There are 50 points possible on this exam. Take note that the problems are not weighted equally.

• When space is provided, show work that justifies your answer. For problems that ask you to prove something, you are allowed to use in your proof any result from the lectures or the relevant sections of the textbook.

• No books, notes, calculators, or other aids are permitted.

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1. For each statement, circle whether it is true or false. (1 point each)

   (a) On the set \( X = \mathbb{R} \), the relation \( x \sim y \iff x - y \in \mathbb{Z} \) is an equivalence relation.  
      True  False

   (b) On the set \( X = \mathbb{R} \), the relation \( x \sim y \iff |x - y| \leq 1 \) is an equivalence relation.  
      True  False

   (c) On the set \( X = \mathbb{R} \), the relation \( x \sim y \iff x^3 - y^3 = x^2 - y^2 \) is an equivalence relation.  
      True  False

   (d) If \( X = G \) is a group, and \( a \in G \) is an element, then the relation \( x \sim y \iff (\exists n \in \mathbb{Z})(x = ya^n) \) is an equivalence relation.  
      True  False

2. For each statement, circle whether it is true or false. (1 point each)

   (a) The subgroup \( \{e, (12)\} \leq S_3 \) is normal.  
      True  False

   (b) If \( G \) is a group, \( N \triangleleft G \), and \( H \leq G \), then necessarily \( N \cap H \triangleleft H \).  
      True  False

   (c) In the group \( S_3 \times S_3 \), the subgroup \( D = \{(\sigma, \sigma) \mid \sigma \in S_3\} \) is normal.  
      True  False

   (d) If \( N \) and \( A \) are groups, and \( \alpha : A \to \text{Aut}(N) \) is a homomorphism, then the semidirect product \( N \rtimes \alpha A \) necessarily contains a normal subgroup isomorphic to \( A \).  
      True  False

3. For each statement, circle whether it is true or false. (1 point each)

   (a) The groups \( \mathbb{Z}_6 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_3 \) are isomorphic.  
      True  False

   (b) The group \( \mathbb{Z}_4 \) is isomorphic to a quotient \( \mathbb{Z}_{10}/N \) of \( \mathbb{Z}_{10} \).  
      True  False

   (c) There is a surjective homomorphism from \( S_5 \) to \( \mathbb{Z}_2 \).  
      True  False

   (d) The group \( \text{SL}(2, \mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \) has a quotient that is a finite group.  
      True  False
4. Let $G$ be a finite group, and let $N \triangleleft G$ be a normal subgroup.

(a) Prove that the order of $gN$ in the quotient $G/N$ is equal to the smallest $n > 0$ such that $g^n \in N$. (3 points)

Since $(gN)^n = g^nN$, we have $(gN)^n = N$ iff $g^nN = N$ iff $g^n \in N$. The order of $gN$ in $G/N$ is the smallest $n > 0$ such that $(gN)^n = N$, so it is also the smallest $n > 0$ such that $g^n \in N$.

(b) Prove that the order of $gN$ in $G/N$ is less than or equal to the order of $g$ in $G$. (3 points)

Let $o(g)$ be the order of $g$ in $G$. Let $o(gN)$ be the order of $gN$.

Then $g^{o(g)} = e \in N$. Since the order of $gN$ in $G/N$ is the smallest $n > 0$ such that $g^n \in N$, we must have $o(gN) \leq o(g)$.

(c) Give an example to show that the order of $gN$ in $G/N$ may be strictly less than the order of $g$ in $G$. (3 points)

Let $G = \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, $g = [1]$, $o(g) = 4$.

$N = \{[0], [2]\} = 2\mathbb{Z}/4\mathbb{Z}$.

Then $G/N = (\mathbb{Z}/4\mathbb{Z})/(2\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

And any element has order at most 2.

So $o(gN) \leq 2 < 4 = o(g)$. 

3
5. For a group $G$, let $Z(G)$ denote the center of $G$, namely the set of elements that commute with every other element:

$$Z(G) = \{g \in G \mid (\forall h \in G)(gh = hg)\}.$$

If $G_1$ and $G_2$ are groups, prove that $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$. (4 points)

$$\begin{align*}
(g_1, g_2)(h_1, h_2) &= (h_1, h_2)(g_1, g_2) \\
\iff (g_1, h_1, g_2, h_2) &= (h_1, g_1, h_2, g_2) \\
\iff g_1 h_1 = h_1 g_1 \text{ and } g_2 h_2 = h_2 g_2
\end{align*}$$

So $(g_1, g_2)$ commutes with everything in $G_1 \times G_2$.

$$\begin{align*}
Z(G_1 \times G_2) &= \{ (g_1, g_2) \mid g_1 \in Z(G_1), g_2 \in Z(G_2) \} \\
&= Z(G_1) \times Z(G_2).
\end{align*}$$

6. Prove that if $H$ and $K$ are finite subgroups of $G$ whose orders are relatively prime then $H \cap K = \{e\}$. (4 points)

\begin{align*}
H \cap K \leq H \text{ so } |H \cap K| &\mid |H| \text{ by Lagrange's theorem.} \\
H \cap K \leq K \text{ so } |H \cap K| &\mid |K|
\end{align*}

\therefore \quad |H \cap K| = \gcd(|H|, |K|) = 1. \text{ So } |H \cap K| = 1

\text{ and } H \cap K = \{e\}.

7. Determine the last digit of $3^{399}$. Hint: Use Euler’s theorem. (4 points)

"Last digit": need to find $3^{399} \text{ mod } 10$.

\gcd(3, 10) = 1, \text{ so we know } 3^{\varphi(10)} \equiv 1 \text{ mod } 10

where $\varphi(10) = |\mathbb{Z}_{10}^*| = 4$ is the number of invertible classes mod 10.

$\mathbb{Z}_{10}^* = \{[1], [3], [7], [9]\}$

Reduce $3^{99}$ mod 4: $3^{99} \equiv (-1)^{99} \equiv -1 \equiv 3 \text{ mod } 4$

so $3^{99} = 4k + 3$ and $3^{3^{99}} = 3^{4k+3} \equiv 3^3 = 27 \equiv 7 \text{ mod } 10$

The last digit is 7.
8. In this problem $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \neq 0\}$ is a group under multiplication, $\mathbb{R}$ is a group under addition, and $\{\pm 1\} = \{1, -1\}$ is a group under multiplication.

(a) Define $\psi : \mathbb{R}^* \to \mathbb{R}$ by $\psi(x) = \ln|x|$. Prove that $\psi$ is a surjective homomorphism and describe the kernel of $\psi$. (3 points)

\[
\text{Homomorphism: } \psi(xy) = \ln|xy| = \ln|x||y| = \ln|x| + \ln|y| = \psi(x) + \psi(y)
\]

\[
\text{Surjective: any } z \in \mathbb{R} \text{ is } z = \ln|e^z|.
\]

\[
\text{Kernel } = \{x \in \mathbb{R}^* \mid \ln|x| = 0\} = \{x \in \mathbb{R}^* \mid |x| = 1\} = \{1, -1\}.
\]

(b) Define $\varphi : \mathbb{R}^* \to \{\pm 1\}$ by $\varphi(x) = x/|x|$. Prove that $\varphi$ is a surjective homomorphism and describe the kernel of $\varphi$. (3 points)

\[
\text{Homomorphism: } \varphi(xy) = \frac{xy}{|xy|} = \frac{x}{|x|} \cdot \frac{y}{|y|} = \varphi(x) \varphi(y)
\]

\[
\text{Surjective: } 1 = \frac{1}{1} = \varphi(1) \quad -1 = \frac{-1}{-1} = \varphi(-1)
\]

\[
\text{Kernel } = \{x \in \mathbb{R}^* \mid \frac{x}{|x|} = 1\} = \{x \in \mathbb{R}^* \mid x = 0\} = \{1, -1\}.
\]

(c) Prove that $\mathbb{R}^*$ is isomorphic to the direct product $\mathbb{R} \times \{\pm 1\}$. (3 points)

Define $\alpha : \mathbb{R}^* \to \mathbb{R} \times \{\pm 1\}$ by $\alpha(x) = (\ln|x|, \frac{x}{|x|})$. The calculations in (a) and (b) show this is a homomorphism.

$\alpha$ is surjective: $(z, s) = \alpha(se^z)$ where $z \in \mathbb{R}$, $s \in \{\pm 1\}$ are anything.

\[
\ker \alpha = \{x \mid \ln|x| = 0 \text{ and } \frac{x}{|x|} = 1\} = \{x \mid x \in \{\pm 1\} \text{ and } x > 0\} = \{1\}
\]

so $\alpha$ is injective also.
9. Let $G$ be the group of upper triangular invertible $2 \times 2$ matrices:

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x \in \mathbb{R}^*, y \in \mathbb{R}, z \in \mathbb{R}^* \right\}.$$

Prove that $G$ is isomorphic to a semidirect product $\mathbb{R} \rtimes_\alpha (\mathbb{R}^* \times \mathbb{R}^*)$, where $\alpha : \mathbb{R}^* \times \mathbb{R}^* \to \text{Aut}(\mathbb{R})$ is a certain homomorphism (you need not specify $\alpha$, you only need to prove it exists). (8 points)

Define $N = \left\{ \begin{bmatrix} 0 & y \\ 0 & z \end{bmatrix} : y \in \mathbb{R} \right\}$

We claim this is a normal subgroup of $G$.

$$\begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_1+y_2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & y_1+y_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -y_1 \\ 0 & 1 \end{bmatrix}$$

so $N$ is a subgroup.

To see $N$ is normal, take $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in G$.

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{-1} & -y/xz \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x & xw+y \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} x^{-1} & -y/xz \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & xw/z \\ 0 & 1 \end{bmatrix} \in N$$

Define $A = \left\{ \begin{bmatrix} x & 0 \\ 0 & z \end{bmatrix} : x \in \mathbb{R}^*, z \in \mathbb{R}^* \right\}$

We claim this is a subgroup.

$$\begin{bmatrix} x_1 & 0 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} x_2 & 0 \\ 0 & z_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 & 0 \\ 0 & z_1z_2 \end{bmatrix}$$

$$\begin{bmatrix} x & 0 \\ 0 & z \end{bmatrix}^{-1} = \begin{bmatrix} x^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix}$$

so it is a subgroup.

$NNA = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is trivial.

Also $NA = G$, since $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 1 & y/z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & z \end{bmatrix}$ (any $x, y, z \in \mathbb{R}$, $y \neq 0$).
So by the recognition theorem for semi-direct products

\[ G = \mathbb{N} \rtimes A \text{ for some } \alpha : A \to \text{Aut}(\mathbb{N}). \]

We claim \( \mathbb{N} \cong \mathbb{R} \) and \( A \cong \mathbb{R}^* \times \mathbb{R}^* \)

Define \( \varphi : \mathbb{R} \to \mathbb{N} \) by \( \varphi(y) = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \) clearly bijective

\[ \varphi(y_1 + y_2) = \begin{bmatrix} 1 & y_1 + y_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y_2 \\ 0 & 1 \end{bmatrix} = \varphi(y_1) \varphi(y_2) \]

so \( \mathbb{N} \cong \mathbb{R} \).

Define \( \psi : \mathbb{R}^* \times \mathbb{R}^* \to A \) by \( \psi(x, z) = \begin{bmatrix} x & 0 \\ 0 & z \end{bmatrix} \) clearly bijective.

\[ \psi((x_1, z_1)(x_2, z_2)) = \psi((x_1, z_1, z_1 z_2)) = \begin{bmatrix} x_1 x_2 & 0 \\ 0 & z_1 z_2 \end{bmatrix} \]

\[ = \begin{bmatrix} x_1 & 0 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} x_2 & 0 \\ 0 & z_2 \end{bmatrix} = \psi(x_1, z_1) \psi(x_2, z_2) \]

so \( A \cong \mathbb{R}^* \times \mathbb{R}^* \)

Thus \( G \cong \mathbb{N} \rtimes A \cong \mathbb{R}^* \ltimes (\mathbb{R}^* \times \mathbb{R}^*). \)
Use this page for work that does not fit on other pages.