Lecture 2

Last lecture: the set of symmetries of $\mathbb{R} \times \mathbb{R}^3$ forms a "group." Now look at another fundamental example.

2) **Permutations**: reordering a set of objects.

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
1 & 2 & 3 & 4 \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\bullet & \bullet & \bullet & \bullet \\
1 & 2 & 3 & 4 \\
\end{array}
\begin{array}{c}
1 \mapsto 2 \\
2 \mapsto 3 \\
3 \mapsto 1 \\
4 \mapsto 4 \\
\end{array}
\]

Definition: A **permutation** of a finite set $F$ is a bijection $\pi: F \to F$.

Recall: A function $f: X \to Y$ is a bijection $\iff$ there is an inverse function $f^{-1}: Y \to X$

$\iff f$ is both injective (one-to-one) and surjective (onto)

i) $f$ is **injective** iff $(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$

ii) $f$ is **surjective** iff $(\forall y \in Y, \exists x \in X \text{ s.t. } f(x) = y)$
Useful fact: When $F = X = Y$ is a finite set, a function $\pi : F \to F$ is bijective $\iff$ it is surjective $\iff$ it is injective.

For any finite set $F$, we can always number the elements $1, 2, \ldots, n = |F|$. In studying permutations we might as well assume that $F = \{1, 2, 3, \ldots, n\}$.

Then a notation for a bijection $\pi : F \to F$ is

$\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix}$ where $a_i \in \{1, 2, \ldots, n\}$

This means that $\pi(1) = a_1$, $\pi(2) = a_2$, $\pi(i) = a_i$ and so on.

Fact: Composition of bijections is a bijection.

(composition of permutations is a permutation)

Example:

$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ $\pi_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

$\pi_2 \circ \pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$
Cycle notation: if $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 1$, we could write $\underline{1 \rightarrow 2 \rightarrow 3}$ a cycle of length 3.

We write $(123)$ for this cycle. This is another notation for permutations.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & \end{pmatrix} = (123)(4) = (123) = (231) = (312)$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & \end{pmatrix} = (13)(24) = (31)(42) = (24)(13)$$

Notation is not unique!

$$\pi_2 \circ \pi_1 = (13)(24)(123) = (142)(3) = (142) \leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & \end{pmatrix}$$

Another such problem: $n = 5$.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (12345)$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = (124)(35)$$

$$\pi_2 \circ \pi_1 = (124)(35)(12345) = (143)(25) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

$$\pi_1 \circ \pi_2 = (12345)(124)(35) = (13)(254) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$

Composition of permutations is not commutative! $\pi_1 \circ \pi_2 \neq \pi_2 \circ \pi_1$, sometimes.
Definition: Denote by $S_n$ the set of permutations of $F = \{1, 2, \ldots, n\}$.

Lemma: The number of elements of $S_n$ is $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$.

Proof: $(1 \ 2 \ 3 \ \cdots \ n)$ $n$ choices for $a_1$,

$\begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix}$ $n-1$ choices for $a_2$,

$n-2$ choices for $a_3$,

$\vdots$

$1$ choice for $a_n$.

In total we have $n(n-1)(n-2)\cdots 2 \cdot 1$ possibilities.

Example: $S_3$ has 6 elements.

$S_3 = \{I, (12), (13), (23), (123), (321)\}$

Note: $(21) = (12)$

$(123) = (231) = (312)$

$(31) = (13)$

$(321) = (213) = (132)$

$(32) = (23)$
Special types of permutations in $S_n$:

(i) Identity permutation: $I = (1)(2)\cdots(n)$ does nothing.

(ii) Transposition: $\pi = (ab)$, $a \mapsto b \mapsto a$.

Swaps $a$ and $b$ and that's all.

(iii) Cycle of length $k$: $\pi = (a_1a_2\cdots a_k)$

\[
\begin{array}{cccc}
  a_1 & \mapsto & a_2 & \mapsto \cdots & \mapsto & a_k \\
\end{array}
\]

Cyclically permutes $a_1, a_2, \ldots, a_k$ and that's all.

Note: 1-cycle = Identity
2-cycle = transposition.

Example: In $S_3$, there are only cycles.

In $S_4$, there are other elements:

\[
(12)(24), \ (14)(23), \ (12)(34)
\]

In $S_5$, can also have

\[
(12)(345), \ etc.
\]

Consider two cycles $(a_1a_2\cdots a_k), (b_1b_2\cdots b_k)$.

They are disjoint if none of the $a$'s equals any of the $b$'s.

Eg. $(123)$ and $(456)$ are disjoint.

$(142)$ and $(35)$ are disjoint.

$(123)$ and $(345)$ are not disjoint (b/c 3).
Proposition 1: Every $\pi \in S_n$ can be written as a product of disjoint cycles, in a way that is essentially unique (unique up to order of the factors).

Proof: $\pi = (1 \ 2 \ \cdots \ n) \ 
\begin{pmatrix} a_1 \ a_2 \ \cdots \ a_n \end{pmatrix}$

Look at sequence $1, \pi(1), \pi^2(1) = \pi(\pi(1)), \pi^3(1), \ldots$
 Eventually the sequence comes back to 1: $\pi^k(1) = 1$ (choose smallest such $k$)
 So $\pi$ contains the $k$-cycle $(1 \ \pi(1) \ \pi^2(1) \ \cdots \ \pi^{k-1}(1))$

Next choose some $a \in \{1, \ldots, n\}$ that does not appear so far, and consider $a, \pi(a), \pi^2(a), \ldots$.
 Eventually this comes back to $a$: $\pi^l(a) = a$.
 So $\pi$ contains the $l$-cycle $(a \ \pi(a) \ \pi^2(a) \ \cdots \ \pi^{l-1}(a))$

Keep repeating this process until all elements $a \in \{1, \ldots, n\}$ have been accounted for.

Example: $(124)(5432) = (125)(34)$

Note: If $\pi_1$ and $\pi_2$ are disjoint cycles, then $\pi_1 \pi_2 = \pi_2 \pi_1$, (they commute)

Eg. $(125)(34) = (34)(125)$
Proposition 2: Every $\pi \in S_n$ can be written as a product of transpositions (in several ways). For a given $\pi$, the number of transpositions appearing in such a factorization is always either even or odd.

Proof of first part: $\pi$ can be written as a product of cycles by Prop. 1, so we just need to show that a cycle can be written as a product of transpositions.

Look at:

$$(a_1a_2\cdots a_k) = (a_{k-1}a_k)(a_{k-2}a_k)\cdots(a_2a_k)(a_1a_k)$$

Proof of second part is omitted. □

Example:

$\pi = (124)(5432) = (24)(14)(32)(42)(52)$

$= (125)(34) = (25)(15)(34)$

$= (125)(324) = (25)(15)(24)(34)$

$q = 5\text{ transp.}$

$= 3\text{ transp.}$ Permutation.

$\pi = 4\text{ transp.}$ Even permutation.

Definition: the sign of a permutation $\pi$ is

$$\text{sgn}(\pi) = \begin{cases} +1 & \pi \text{ is even} \\ -1 & \pi \text{ is odd} \end{cases}$$
Example: \[ \text{sgn}(I) = 1 \quad \text{sgn}(\text{cab}) = -1 \]

\[ \text{sgn}(\{a_1, a_2, \ldots, a_k\}) = (-1)^{k-1} \]

If \( \pi = (a_1 \ldots a_k) \) is a cycle, the inverse is \( \pi^{-1} = (a_k a_{k-1} \ldots a_2 a_1) \).

E.g. \( \pi = (4215) \Rightarrow \pi^{-1} = (5124) = (1245) \).

If \( \pi \) is a product of cycles, then \( \pi^{-1} \) is the product of the inverse cycles in the reverse order.

\[
\pi = (a_1 \ldots a_k)(b_1 \ldots b_l) \ldots (z_1 \ldots z_r)
\]
\[
\pi^{-1} = (z_r \ldots z_1) \ldots (b_l \ldots b_1)(a_k \ldots a_1)
\]

\[
\pi \circ \pi^{-1} = (a_1 \ldots a_k)(b_1 \ldots b_l) \ldots (z_1 \ldots z_r)(z_r \ldots z_1) \ldots (b_l \ldots b_1)(a_k \ldots a_1)
\]

Thus \( S_n \) is a group. \( S_n = \{ \text{permutations of } \{1, \ldots, n\} \} \)

operation = composition, which is

(i) Associative: \( (\pi_1 \circ \pi_2) \circ \pi_3 = \pi_1 \circ (\pi_2 \circ \pi_3) \)

(ii) has Identity: \( \pi \circ \text{I} = \text{I} \circ \pi = \pi \)

(iii) has Inverses: \( \pi \circ \pi^{-1} = \pi^{-1} \circ \pi = \text{I} \).