Examples of semi-direct products

Example \( \text{Aff}(\mathbb{R}^n) = \{ T_{M,b} | M \in \text{GL}(n,\mathbb{R}), b \in \mathbb{R}^n \} \)

where \( T_{M,b}(x) = Mx + b \).

Let \( N = \text{Trans}(\mathbb{R}^n) = \{ T_{I,b} | b \in \mathbb{R}^n \} \quad T_{I,b}(x) = x + b \)

and \( A = \{ T_{M,I} | M \in \text{GL}(n,\mathbb{R}) \} \quad T_{M,I}(x) = Mx \)

Then \( N \leq \text{Aff}(\mathbb{R}^n) \quad A \leq \text{Aff}(\mathbb{R}^n) \)

\( N \) is normal: \( T_{M,b} T_{I,c} T_{M,b}^{-1} = T_{I,Mc} \)

\( NA = \text{Aff}(\mathbb{R}^n) \) for \( T_{M,b} = T_{I,b} \circ T_{M,I} \)

\( \in N \quad \in A \)

And \( N \cap A = \{ T_{I,b} \} \) which is the trivial subgroup.

The map \( \alpha: A \to \text{Aut}(N) \) is the conjugation homomorphism:

\[ \alpha_{T_{M,I}}(T_{I,b}) = T_{I,Mb} \]

By the proposition, we see that \( N \cong A \leq \text{Aff}(\mathbb{R}^n) \)

The group \( N \) is isomorphic to \( (\mathbb{R}^n,+) \), while \( A \) is isomorphic to \( \text{GL}(n,\mathbb{R}) \)

the homomorphism \( \alpha: A \to \text{Aut}(N) \) corresponds to

\[ \alpha: \text{GL}(n,\mathbb{R}) \to \text{Aut}(\mathbb{R}^n) \]

\[ \alpha_M(b) = Mb \]

So we can also say \( \mathbb{R}^n \times \text{GL}(n,\mathbb{R}) \cong \text{Aff}(\mathbb{R}^n) \).
Another example: Let \( N = \mathbb{Z}_7 \). There is an automorphism 
\[ \varphi : \mathbb{Z}_7 \rightarrow \mathbb{Z}_7 \quad \varphi([k]) = [2k] \]
\[ \varphi^2([k]) = [4k] \quad \varphi^3([k]) = [8k] = [k] \]
Thus \( \varphi \) has order 3 in \( \text{Aut}(\mathbb{Z}_7) \).

Then there is a homomorphism
\[ \alpha : \mathbb{Z}_3 \rightarrow \text{Aut}(\mathbb{Z}_7) \]
\[ \alpha([k]) = \varphi^k \]
and we can form the semi-direct product \( \mathbb{Z}_7 \rtimes \mathbb{Z}_3 \).

This is a non-abelian group with 21 elements. Let's do some calculations.
\[ ([3], [1])([4], [0]) \]
\[ = ([3] + \alpha_{[1]}([4]), [1]+[0]) \]
\[ = ([3] + \varphi([4]), [1]) \]
\[ = ([3] + 2[4], [1]) \]
\[ = ([3] + [1], [1]) \]
\[ = ([4], [1]) \]

More generally \( ([n]_7, [a]_3)([n']_7, [a']_3) \)
\[ = ([n]_7 + \alpha_{[a]_3}([n']_7), [a]_3 + [a']_3) \]
\[ = ([n]_7 + [2^n]_7, [a+a']_3) \]
\[ = ([n+2^n]_7, [a+a']_3) \)
Group actions

"Group" is an abstract concept, but many examples are "Groups of symmetry transformations" like $\text{Sym}(X)$, $\text{D}_n$, $\text{GL}(n, \mathbb{R})$, and so on.

Given a group $G$, it is then natural to ask if we can think of $G$ as symmetries of something (in an abstract sense).

Let $X$ be a set, and let $G$ be a group.

**Definition:** An action of $G$ on $X$ is a function

$$G \times X \rightarrow X \text{ denoted } (g, x) \mapsto g \cdot x$$

satisfying

(i) $e \cdot x = x$ for all $x \in X$, where $e$ is the identity element of $G$.

(ii) $g \cdot (h \cdot x) = (gh) \cdot x$

for all $g, h \in G$, $x \in X$.

There is another way to think about actions, which is as a homomorphism $\alpha : G \rightarrow \text{Sym}(X)$, where $\text{Sym}(X) = \{f : X \rightarrow X \mid f \text{ bijective}\}$ is the symmetric group of $X$.

**Lemma** An action $G \times X \rightarrow X$ of a group $G$ on a set $X$ determines and is determined by a homomorphism

$\alpha : G \rightarrow \text{Sym}(X)$

$g \mapsto \alpha_g$

where $\alpha_g(x) = g \cdot x$
Proof. Let \( G \times X \to X \) \((g, x) \mapsto g \cdot x\) be a group action.

For each \( g \in G \), let \( \alpha_g : X \to X \) be the function \( \alpha_g(x) = g \cdot x \). We claim \( \alpha_g \) is bijective. In fact, its inverse is \( \alpha_g^{-1} \) for \( \alpha_g^{-1}(\alpha_g(x)) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x \) by axioms of a group action.

So for all \( g \in G \), \( \alpha_g \in \text{Sym}(X) \), and \( \alpha : G \to \text{Sym}(X) \) is a function.

Lastly, we check \( \alpha \) is a homomorphism:

\[
\alpha_{gh}(x) = (gh) \cdot x = g \cdot (h \cdot x) = \alpha_g(\alpha_h(x)) \quad (\text{all } x \in X)
\]

so \( \alpha_{gh} = \alpha_g \circ \alpha_h \), as desired.

* Conversely, suppose \( \alpha : G \to \text{Sym}(X) \) is a homomorphism.

Define a function \( G \times X \to X \) by declaring \( g \cdot x = \alpha_g(x) \).

We must check the axioms of a group action.

(i) Since \( \alpha \) is a homomorphism, it take identity to identity.

Thus \( \alpha_e = \text{Id}_X \) where \( \text{Id}_X : X \to X \) is the identity function.

So \( e \cdot x = \alpha_e(x) = \text{Id}_X(x) = x \), as desired.

(ii) Since \( \alpha \) is a homomorphism, \( \alpha_{gh} = \alpha_g \circ \alpha_h \)

so \( (gh) \cdot x = \alpha_{gh}(x) = \alpha_g(\alpha_h(x)) = g \cdot (h \cdot x) \), as desired.

Due to this lemma, we will often switch between the two "pictures" of a group action: (a) A map \( G \times X \to X \)

(b) A map \( G \to \text{Sym}(X) \)

Definition 5.1.1 in the book corresponds to picture (b).
Examples:

1. \( X \) any set, \( G = \text{Sym}(X) \) symmetric group.
   
   \( \text{Sym}(X) \) acts on \( X \).
   
   \( \text{Sym}(X) \times X \rightarrow X \)
   
   \( (\sigma, x) \rightarrow \sigma(x) \) (apply function \( \sigma \) to \( x \))

   The corresponding homomorphism \( \kappa : \text{Sym}(X) \rightarrow \text{Sym}(X) \)
   
   is the identity.

2. Let \( H \leq \text{Sym}(X) \) be a subgroup. Then \( H \) acts on \( X \)
   
   similarly to example 1.

3. \( X = \mathbb{R}^n, \ G = \text{GL}(n, \mathbb{R}) \)
   
   \( \text{GL}(n, \mathbb{R}) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)
   
   \( (A, x) \rightarrow Ax \) multiply vector and matrix.

   Also any subgroup of \( \text{GL}(n, \mathbb{R}) \) acts on \( \mathbb{R}^n \).

4. Given a group \( G \), we can take \( X = G \), and find several
   
   actions of \( G \) on itself.

4a. Recall left multiplication \( L_g : G \rightarrow G \)
   
   \( L_g(h) = gh \).

   We saw in lecture 10 that this gives a homomorphism
   
   \( L : G \rightarrow \text{Sym}(G) \)
   
   \( g \rightarrow L_g \)

   So this is a group action of \( G \) on \( G \), called left multiplication.

   (The corresponding map \( G \times G \rightarrow G \) is just multiplication.

4b. For \( g \in G \), we have conjugation by \( g : C_g : G \rightarrow G \)
   
   \( C_g(h) = g h g^{-1} \). We saw in lecture 22

   that the function \( C : G \rightarrow \text{Aut}(G) \), \( g \rightarrow C_g \)

   is a homomorphism. Now \( \text{Aut}(G) \) is a subgroup of \( \text{Sym}(G) \),

   so we can also regard conjugation as a homomorphism.

   \( C : G \rightarrow \text{Sym}(G) \)

   This is called the conjugation action.

   \( G \times G \rightarrow G \)

   \( (g, h) \mapsto g h g^{-1} \)
What about right multiplication? \( \text{Rg: } G \to G \quad \text{Rg}(u) = uh \)

This does define a function \( G \to \text{Sym}(G) \)

\[ g \mapsto \text{Rg} \]

But unless \( G \) is abelian, this function is NOT A HOMOMORPHISM.

For \( \text{Rg}(x) = xgh = Rh \cdot \text{Rg}(x) \), so \( \text{Rg} = Rh \circ \text{Rg} \),

and \( \text{Rg} \not\equiv \text{Rg} \circ Rh \) unless \( gh = hg \).

On the other hand, \( \alpha: G \to \text{Sym}(G) \quad \alpha(x) = xg^{-1} \)

\[ \alpha_g = \text{Rg}^{-1} \]

is a homomorphism!

\[ \alpha_{gh}(x) = x(gh)^{-1} = xh^{-1}g^{-1} = \text{Rg}(Rh(x)) = \alpha_g(\alpha_h(x)) \]

So "right multiplication by the inverse" is a group action of \( G \) on \( G \).

Some basic sets associated to a group action.

**Definition:** Let \( G \times X \to X \) be a group action.

1. For \( x \in X \), the set \( G \cdot x = \{ gx | g \in G \} \) is called the orbit of \( x \) (the book uses \( \Theta(x) \) for this.)

2. The action is **transitive** if there is an \( x \in X \) such that \( G \cdot x = X \).

3. For \( x \in X \), the stabilizer of \( x \) is \( \text{Stab}(x) = \{ g \in G | gx = x \} \)

   it is a subset of \( G \).

4. The **kernel** of the action is \( \ker(\alpha: G \to \text{Sym}(X)) \)

   \[ \text{it equals } \bigcap_{x \in X} \text{Stab}(x) \]