Math 481: Final Exam Solutions
Spring 2020

Instructions:

• There are 95 points possible on this exam. Take note that the problems are not weighted equally.

• Please complete the problems and upload your solutions to Box, just as you would do with a homework assignment.

• For problems that ask you to prove something, you are allowed to use in your proof any result from the lectures or the relevant sections of the textbook.

• For this take-home exam you may refer to your notes, the lecture videos, homework solutions, and any of resources the mentioned on the course website. You may not refer to online sources not mentioned on the course website (e.g. google searches), and you may not discuss the exam with other students.

• If you have questions (for example, needing clarification of a problem statement). Please email the instructor at jpascale@illinois.edu.

• This exam will be posted online no later than 7:00am CDT on Monday, May 11, and your solutions should be uploaded by 11:59pm CDT on Friday, May 15.
1. (7 points) Show that the function \( f : \mathbb{R} \to \mathbb{R}, \)
\[
f(x) = \begin{cases} 
  e^{-1/x}, & x > 0, \\
  0, & x \leq 0,
\end{cases}
\]
is smooth. (If you cannot do this, you can get 5 points for showing that \( f, f', f'', f''' \) are all continuous.)

**Solution:** A “polynomial in \( x^{-1} \)” is a function of the form

\[
q(x) = a_n x^{-n} + a_{n-1} x^{-n+1} + \cdots + a_1 x^{-1} + a_0
\]

(a linear combination of non-positive powers of \( x \)).

We first prove the following lemma: If \( q(x) \) is a polynomial in \( x^{-1} \), then the derivative of \( e^{-1/x} q(x) \) is also \( e^{-1/x} \) times a polynomial in \( x^{-1} \). Proof: we have

\[
\frac{d}{dx} (e^{-1/x} q(x)) = e^{-1/x} x^{-2} q(x) + e^{-1/x} q'(x) = e^{-1/x} (x^{-2} q(x) + q'(x))
\]

Since \( q(x) \) is a polynomial in \( x^{-1} \), \( q'(x) \) is also a polynomial in \( x^{-1} \), and so \( x^{-1} q(x) + q'(x) \) is a polynomial in \( x^{-1} \).

Using the lemma and induction, we see that all of the derivatives of \( e^{-1/x} \) are of the form \( e^{-1/x} \) times a polynomial in \( x^{-1} \):

\[
\frac{d^n}{dx^n} (e^{-1/x}) = e^{-1/x} q_n(x)
\]

where \( q_n(x) \) is a polynomial in \( x^{-1} \). We now claim that

\[
\lim_{x \to 0^+} e^{-1/x} q_n(x) = 0
\]

By changing variables \( u = 1/x \), we are claiming that

\[
\lim_{u \to \infty} e^{-u} q_n(1/u) = 0
\]

Now \( q_n(1/u) \) is a ordinary polynomial in \( u \), so it suffices to prove

\[
\lim_{u \to \infty} e^{-u} u^k = 0
\]

For every \( k \). We prove this by induction on \( k \). For \( k = 0 \), we are saying \( \lim_{u \to 0} e^{-u} = 0 \), which is true. Suppose the statement is true for \( k - 1 \), then by L'Hopital’s rule:

\[
\lim_{u \to \infty} e^{-u} u^k = \lim_{u \to \infty} \frac{u^k}{e^u} = \lim_{u \to \infty} \frac{k u^{k-1}}{e^u} = 0
\]

where in the last step we use the induction hypothesis.
Now we prove that $f$ is smooth. It is clearly smooth on the domains $x < 0$ where it is identically 0 and $x > 0$ where it is given by the formula $e^{-1/x}$. The only question is whether it is smooth at 0. For this we need that the left and right limits of the derivatives of the two formulas agree

$$\lim_{x \to 0^-} \frac{d^n}{dx^n}(0) = \lim_{x \to 0^+} \frac{d^n}{dx^n}(e^{-1/x})$$

which is exactly what has been shown above.

2. (7 points) Recall $GL(2)$, the set of invertible $2 \times 2$ matrices:

$$GL(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc \neq 0 \right\}$$

There is a smooth map $F : GL(2) \to GL(2)$ that takes a matrix to its inverse: $F(A) = A^{-1}$. If $I$ denotes the identity matrix, then $F(I) = I$, so the derivative of $F$ at $I$ defines a map

$$F_* : T_I GL(2) \to T_I GL(2).$$

Using $\left\{ \frac{\partial}{\partial a} \bigg|_I, \frac{\partial}{\partial b} \bigg|_I, \frac{\partial}{\partial c} \bigg|_I, \frac{\partial}{\partial d} \bigg|_I \right\}$ as a basis for $T_I GL(2)$, compute $F_*$. 

**Solution:** The formula for $F$ is

$$F\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & -b \\ ad - bc & -c \end{bmatrix}$$

Taking the derivative of this matrix-valued function with respect to each variable gives

$$\frac{\partial}{\partial a} F\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \frac{-d}{(ad - bc)^2} & \frac{bd}{(ad - bc)^2} \\ \frac{cd}{(ad - bc)^2} & \frac{-c}{(ad - bc)^2} \end{bmatrix}$$

$$\frac{\partial}{\partial b} F\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \frac{d}{(ad - bc)^2} & \frac{(ad - bc)(-1) - (b)(-c)}{(ad - bc)^2} \\ \frac{-c}{(ad - bc)^2} & \frac{-c}{(ad - bc)^2} \end{bmatrix}$$

$$\frac{\partial}{\partial c} F\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \frac{-bd}{(ad - bc)^2} & \frac{-b^2}{(ad - bc)^2} \\ \frac{(ad - bc)(-1) - (c)(b)}{(ad - bc)^2} & \frac{b}{(ad - bc)^2} \end{bmatrix}$$

$$\frac{\partial}{\partial d} F\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} \frac{(ad - bc) - ad}{(ad - bc)^2} & \frac{ab}{(ad - bc)^2} \\ \frac{ac}{(ad - bc)^2} & \frac{-b}{(ad - bc)^2} \end{bmatrix}$$

Then plugging in $a = 1, b = 0, c = 0, d = 1$ gives us the derivatives at $I$:

$$\left. \frac{\partial}{\partial a} \right|_I F\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
\[
\frac{\partial}{\partial b} \bigg|_{I} F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}
\]
\[
\frac{\partial}{\partial c} \bigg|_{I} F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}
\]
\[
\frac{\partial}{\partial d} \bigg|_{I} F\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}
\]

This allows us to read off \( F^* \):

\[
F^*\left(\frac{\partial}{\partial a} \bigg|_{I}\right) = -\frac{\partial}{\partial a} \bigg|_{I}
\]
\[
F^*\left(\frac{\partial}{\partial b} \bigg|_{I}\right) = -\frac{\partial}{\partial b} \bigg|_{I}
\]
\[
F^*\left(\frac{\partial}{\partial c} \bigg|_{I}\right) = -\frac{\partial}{\partial c} \bigg|_{I}
\]
\[
F^*\left(\frac{\partial}{\partial d} \bigg|_{I}\right) = -\frac{\partial}{\partial d} \bigg|_{I}
\]

So \( F^* \) takes each basis element to its negative.

3. In \( \mathbb{R}^4 \), consider the set \( Q = \{(w, x, y, z) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 - w^2 = 1\} \).

(a) (6 points) Use the Regular Value Theorem to prove that \( Q \) is a submanifold of \( \mathbb{R}^4 \).

**Solution:** Consider the map \( G : \mathbb{R}^4 \rightarrow \mathbb{R} \) given by

\[
G(w, x, y, z) = x^2 + y^2 + z^2 - w^2
\]

so that \( Q = G^{-1}(1) \). We need to show that 1 is a regular value of \( G \). The derivatives of \( G \) are

\[
[G_*] = \begin{bmatrix} -2w & 2x & 2y & 2z \end{bmatrix}
\]

This matrix has rank 1 unless all entries are zero, in which case \((w, x, y, z) = (0, 0, 0, 0)\); this is the only critical point. But \( G(0, 0, 0, 0) = 0 \), so the critical point is not in \( Q = G^{-1}(1) \). So all points in \( Q \) are regular points, and so 1 is a regular value of \( G \). Thus we concluded that \( Q \) is a submanifold of \( \mathbb{R}^4 \) of dimension \( 4 - 1 = 3 \).

(b) (6 points) Find the critical points and critical values of the function \( g : Q \rightarrow \mathbb{R} \) defined by

\[
g(w, x, y, z) = x^2 + 2y^2 + z^2.
\]
Solution: If $w \neq 0$, then $\frac{\partial G}{\partial w} \neq 0$, so we can solve for $w$ in terms of $(x, y, z)$ and use $(x, y, z)$ as coordinates, namely $w = \pm \sqrt{x^2 + y^2 + z^2 - 1}$. This gives us two coordinate charts $U^\pm$, where $U^+ = \{(w, x, y, z) \in Q \mid w > 0\}$ and $U^-$ is where $w < 0$. The images of the charts are the triples $(x, y, z)$ such that the expression under the square root is positive, which is to say $x^2 + y^2 + z^2 > 1$, the exterior of a circle of radius 1. The critical points of the function $g$ in these charts are where all partial derivatives are zero:

$$2x = 0, \quad 4y = 0, \quad 2z = 0,$$

thus $(x, y, z) = (0, 0, 0)$. But this is not a point in the image of the chart. We conclude that $g$ has no critical points in the charts $U^\pm$, meaning that any critical points (if they exist) must occur at points where $w = 0$.

If $x \neq 0$, we can use $(w, y, z)$ as coordinates via

$$x = \pm \sqrt{1 + w^2 - y^2 - z^2}$$

Again there are two charts for the two possible signs of the square root, and the image of the charts are the set where the expression under the square root is positive. In these coordinates $g$ becomes

$$g = (1 + w^2 - y^2 - z^2) + 2y^2 + z^2 = 1 + w^2 + y^2$$

This has critical points whenever $w = 0, y = 0, z$ is anything, and $x = \pm \sqrt{1 - z^2}$. At any such point, the value of $g$ will be

$$g(0, \sqrt{1 - z^2}, 0, z) = (1 - z^2) + 0^2 + z^2 = 1$$

The case $z \neq 0$ is similar to the previous one. We now have $z = \pm \sqrt{1 + w^2 - x^2 - y^2}$ and $g = x^2 + 2y^2 + (1 + w^2 - x^2 - y^2) = 1 + w^2 + y^2$. The critical points are where $w = 0, y = 0, x$ is anything, and $z = \pm \sqrt{1 - x^2}$. The value of $g$ at such points is again 1.

If $y \neq 0$, then we can write $y = \pm \sqrt{1 + w^2 - x^2 - z^2}$. Then

$$g = x^2 + 2(1 + w^2 - x^2 - z^2) + z^2 = 2 + 2w^2 - x^2 - z^2$$

The critical points are then where $w = 0, x = 0, z = 0, y = \pm 1$. At such a point, $g(0, 0, \pm 1, 0) = 2$.

A more geometric description of the critical points: There is a circle of critical points in the $xz$-plane: $\{w = 0, y = 0, x^2 + z^2 = 1\}$, and all of these points have critical value 1. There are two other critical points $(0, 0, \pm 1, 0)$ with critical value 2.
4. (a) (7 points) In \( \mathbb{R}^3 \) with coordinates \((x^1, x^2, x^3)\) compute the flow of the vector field 

\[
X = 5 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + 2x^3 \frac{\partial}{\partial x^3}
\]

**Solution:** Suppose the solution is 

\[
\varphi_t (x^1, x^2, x^3) = (u_t(x^1, x^2, x^3), v_t(x^1, x^2, x^3), w_t(x^1, x^2, x^3)).
\]

The the differential equations we must solve are 

\[
\frac{du_t}{dt} = 5 \\
\frac{dv_t}{dt} = w_t \\
\frac{dw_t}{dt} = 2w_t
\]

The solution of the first is \(u_t = 5t + C_1\). The solution of the third is \(w_t = C_2e^{2t}\). Then the second becomes 

\[
\frac{dv_t}{dt} = C_2e^{2t}
\]

So \(v_t = (C_2/2)e^{2t} + C_3\). Using the initial conditions \(u_0 = x^1, v_0 = x^2, w_0 = x^3\), we get \(C_1 = x^1, C_2 = x^3\), and \(C_3 = x^2 - x^3/2\). Thus we have 

\[
\varphi_t (x^1, x^2, x^3) = (x^1 + 5t, x^2 + x^3(e^{2t} - 1)/2, x^3e^{2t})
\]

(b) (5 points) Compute \(X(h)\) where \(h\) is the function \(h(x^1, x^2, x^3) = x^1 x^2 \sin(x^3)\)

**Solution:**

\[
\frac{\partial h}{\partial x^1} = x^2 \sin(x^3) \\
\frac{\partial h}{\partial x^2} = x^1 \sin(x^3) \\
\frac{\partial h}{\partial x^3} = x^1 x^2 \cos(x^3)
\]

\(X(h) = 5x^2 \sin(x^3) + x^3 x^1 \sin(x^3) + 2x^3 x^1 x^2 \cos(x^3)\)

(c) (7 points) Let \(Y = \frac{\partial}{\partial x^3}\). Compute the Lie derivative \(L_X Y\). You may use any method.

**Solution:** We know \(L_X Y = [X, Y]\), so we compute the Lie bracket \([X, Y] = XY - YX\). 

\[
XY(f) = \left(5 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + 2x^3 \frac{\partial}{\partial x^3}\right) \left(\frac{\partial}{\partial x^3}\right)(f) = 5 \frac{\partial^2 f}{\partial x^1 \partial x^3} + x^3 \frac{\partial^2 f}{\partial x^2 \partial x^3} + 2x^3 \frac{\partial^2 f}{(\partial x^3)^2}
\]
\[ YX(f) = \left( \frac{\partial}{\partial x^3} \right) \left( 5 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + 2x^3 \frac{\partial}{\partial x^3} \right)(f) = \left( \frac{\partial f}{\partial x^3} \right) \left( 5 \frac{\partial f}{\partial x^1} + x^3 \frac{\partial f}{\partial x^2} + 2x^3 \frac{\partial f}{\partial x^3} \right) \]

\[ = 5 \frac{\partial^2 f}{\partial x^1 \partial x^3} + \frac{\partial f}{\partial x^2} + x^3 \frac{\partial^2 f}{\partial x^2 \partial x^3} + 2 \frac{\partial f}{\partial x^3} + 2x^3 \frac{\partial^2 f}{(\partial x^3)^2} \]

Thus

\[ XY(f) - YX(f) = -\frac{\partial f}{\partial x^2} - 2 \frac{\partial f}{\partial x^3} = \left( -\frac{\partial}{\partial x^2} - 2 \frac{\partial}{\partial x^3} \right)(f) \]

So

\[ L_X Y = [X, Y] = -\frac{\partial}{\partial x^2} - 2 \frac{\partial}{\partial x^3} \]

5. Consider the following subsets of \( \mathbb{R}^3 \):

\[ B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}, \]

\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}, \]

\[ H_+ = \{(x, y, z) \in S^2 \mid z \geq 0\} \]

\[ H_- = \{(x, y, z) \in S^2 \mid z \leq 0\} \]

and let \( \gamma \) be the parameterized curve

\[ \gamma(t) = (\sin t, \cos t, 0). \]

In words: \( B^3 \) is the unit ball, \( S^2 \) is the unit sphere, \( H_+ \) is the upper hemisphere, \( H_- \) is the lower hemisphere, and \( \gamma \) is a parameterization of the equator. Suppose we make the following choices:

- The orientation on \( B^3 \) is taken to be inherited from the standard orientation of \( \mathbb{R}^3 \), meaning that \( \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \) is a positively oriented ordered basis of the tangent spaces.

- The orientation on \( S^2 \) is the boundary orientation as \( S^2 = \partial B^3 \). This orientation restricts to orientations of \( H_+ \) and \( H_- \).

Suppose that \( \alpha \in \Lambda^1(\mathbb{R}^3) \) is a differential form of degree one, and suppose that

\[ \int_{\gamma} \alpha = 10. \]

Compute the following:

(a) \( (4 \text{ points}) \int_{H_+} d\alpha, \)

(b) \( (4 \text{ points}) \int_{H_-} d\alpha, \)
(c) (4 points) $\int_{S^2} d\alpha$.

**Solution:** Consider the orientation of $S^2 = \partial B^3$ at the point $(1,0,0)$. We know that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ is a positive basis for $B^3$, and the outward normal vector to $S^2$ at $(1,0,0)$ is $\frac{\partial}{\partial x}$. By the outward-normal first rule, we discover that $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ is a positive basis for the tangent space to $S^2$ at $(1,0,0)$. Thus it is also a positive basis for $H_+$ and $H_-$. The outward normal to $H_+$ at $(1,0,0)$ is $-\frac{\partial}{\partial z}$, and the outward normal to $H_-$ at $(1,0,0)$ is $\frac{\partial}{\partial z}$.

Now note that $(1,0,0)$ also lies on $\gamma$; it is $\gamma(\pi/2)$. At this point $\dot{\gamma}(\pi/2) = -\frac{\partial}{\partial y}$.

The outward normal to $H_+$ followed by $\dot{\gamma}$ is $\{-\frac{\partial}{\partial z}, -\frac{\partial}{\partial y}\}$, which is not equivalent to $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$, since negating each vector and swapping the order flips the sign of the determinant 3 times. Thus when we apply Stokes’ theorem, we find

$$\int_{H_+} d\alpha = -\int_{\gamma} \alpha = -10$$

The outward normal to $H_-$ followed by $\dot{\gamma}$ is $\{\frac{\partial}{\partial z}, -\frac{\partial}{\partial y}\}$, which is equivalent to $\{\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. So by Stokes’ theorem,

$$\int_{H_+} d\alpha = \int_{\gamma} \alpha = 10$$

For $\int_{S^2} d\alpha$, we can either combine the two previous parts

$$\int_{S^2} d\alpha = \int_{H_+} d\alpha + \int_{H_-} d\alpha = -10 + 10 = 0$$

or we can use Stokes’ theorem again:

$$\int_{S^2} d\alpha = \int_{B^3} d\alpha = \int_{B^3} 0 = 0$$

6. Let $M$ be a manifold, and let $(U, \varphi)$ and $(U, \varphi')$ be two charts on $M$ with the same domain $U$. Then we have two compatible coordinate systems in $U$: the coordinates $(x^1, x^2, \ldots, x^n)$ coming from $\varphi$ and the coordinates $(x'^1, x'^2, \ldots, x'^n)$ coming from $\varphi'$. Since these are coordinate systems on the same set, each $x^i$ is a smooth function of the $x'^j$. We use the notation

$$\frac{\partial x^i}{\partial x'^\alpha}$$
to denote the partial derivatives of one coordinate with respect to the other coordinate system.

Let $g$ be a metric on $M$, and suppose that $g$ has local coordinate representations

$$g = \sum_{i,j=1}^{n} g_{ij} \, dx^i \otimes dx^j = \sum_{i,j=1}^{n} g'_{\alpha\beta} \, dx'^{\alpha} \otimes dx'^{\beta}$$

in the two coordinate systems.

(a) (6 points) Show that

$$g'_{\alpha\beta} = \sum_{i,j=1}^{n} g_{ij} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}}$$

**Solution:** Using the transformation rule for 1-forms under coordinate changes:

$$dx^i = \sum_{\alpha=1}^{n} \frac{\partial x^i}{\partial x'^{\alpha}} \, dx'^{\alpha}$$

So:

$$dx^i \otimes dx^j = \left( \sum_{\alpha=1}^{n} \frac{\partial x^i}{\partial x'^{\alpha}} \, dx'^{\alpha} \right) \otimes \left( \sum_{\beta=1}^{n} \frac{\partial x^j}{\partial x'^{\beta}} \, dx'^{\beta} \right) = \sum_{\alpha,\beta=1}^{n} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \, dx'^{\alpha} \otimes dx'^{\beta}$$

$$g = \sum_{i,j=1}^{n} g_{ij} \, dx^i \otimes dx^j = \sum_{i,j=1}^{n} g_{ij} \sum_{\alpha,\beta=1}^{n} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \, dx'^{\alpha} \otimes dx'^{\beta}$$

By rearranging the summations:

$$g = \sum_{\alpha,\beta=1}^{n} \left( \sum_{i,j=1}^{n} g_{ij} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} \right) \, dx'^{\alpha} \otimes dx'^{\beta}$$

The coefficient of $dx'^{\alpha} \otimes dx'^{\beta}$ in this expression, which by definition is equal to $g'_{\alpha\beta}$, is

$$\sum_{i,j=1}^{n} g_{ij} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}}$$

(b) (6 points) Find the expression for the standard Euclidean metric on $\mathbb{R}^2$ in polar coordinates $(r, \theta)$.

**Solution:** Using $(x, y)$ for the standard cartesian coordinates, we have

$$g = dx \otimes dx + dy \otimes dy$$

Since $x = r \cos \theta$ and $y = r \sin \theta$, we have

$$dx = \cos \theta \, dr - r \sin \theta \, d\theta$$

$$dy = \sin \theta \, dr + r \cos \theta \, d\theta$$

Therefore,

$$dx \otimes dx + dy \otimes dy = (\cos \theta \, dr - r \sin \theta \, d\theta) \otimes (\cos \theta \, dr - r \sin \theta \, d\theta) + (\sin \theta \, dr + r \cos \theta \, d\theta) \otimes (\sin \theta \, dr + r \cos \theta \, d\theta)$$

$$= r^2 (\cos^2 \theta \, dr \otimes dr + \sin^2 \theta \, dr \otimes dr + \sin \theta \cos \theta \, dr \otimes d\theta + \sin \theta \cos \theta \, d\theta \otimes dr)$$

$$= r^2 \, dr \otimes dr + r^2 \sin^2 \theta \, d\theta \otimes d\theta$$

So the standard Euclidean metric in polar coordinates is

$$g = r \, dr \otimes dr + r^2 \sin^2 \theta \, d\theta \otimes d\theta$$
\[ dy = \sin \theta \, dr + r \cos \theta \, d\theta \]

So
\[
\begin{align*}
\frac{dx}{dx} & = \cos^2 \theta \, dr \otimes dr - r \sin \theta \cos \theta \, dr \otimes d\theta - r \sin \theta \cos \theta \, d\theta \otimes dr + r^2 \sin^2 \theta \, d\theta \otimes d\theta \\
\frac{dy}{dy} & = \sin^2 \theta \, dr \otimes dr + r \sin \theta \cos \theta \, dr \otimes d\theta + r \sin \theta \cos \theta \, d\theta \otimes dr + r^2 \cos^2 \theta \, d\theta \otimes d\theta \\
g & = (\cos^2 \theta + \sin^2 \theta) \, dr \otimes dr + r^2 (\sin^2 \theta + \cos^2 \theta) \, d\theta \otimes d\theta \\
g & = dr \otimes dr + \frac{d\theta}{\sin^2 \theta} = \frac{r^2 \cos^2 \theta}{\sin^2 \theta} \, d\theta \otimes d\theta
\end{align*}
\]

(c) (6 points) Show that any one-dimensional Riemannian manifold is flat, that is, near any point there is a local coordinate \( x' \) such that the metric \( g \) takes the form \( g = dx' \otimes dx' \). Hint: start with any coordinate \( x \) and write the metric as \( g = g_{11} dx \otimes dx \). Using part (a), figure out the condition that will make \( g'_{11} = 1 \), and use that condition to solve for \( x' \) as a function of \( x \).

Solution: Following the hint, suppose \( g = g_{11}(x) \, dx \otimes dx \) with respect some coordinate \( x \). The desired coordinate \( x' \) is a function of \( x \), and we wish to ensure that in this coordinate the metric is \( g = dx' \otimes dx' \), meaning \( g'_{11} = 1 \). Thus we must have
\[
1 = g_{11}(x) \frac{dx}{dx'} \frac{dx'}{dx}
\]
Since \((dx/dx')^{-1} = dx'/dx\), this equation can be rewritten
\[
\left( \frac{dx'}{dx} \right)^2 = g_{11}(x)
\]
Since is always positive, we can solve this differential equation as
\[
\frac{dx'}{dx} = \sqrt{g_{11}(x)}
\]
\[
x' = \int \sqrt{g_{11}(x)} \, dx
\]
This defines \( x' \) up to the addition of a constant, which doesn’t matter.

(d) (6 points) Suppose that \( M = \{ x \in \mathbb{R} \mid x > 0 \} \) is given the metric \( g = x^{-2} \, dx \otimes dx \). Find the coordinate \( x' \) such that \( g = dx' \otimes dx' \) as an explicit function of \( x \).

Solution: Following the notation from the previous part, we have \( g_{11}(x) = x^{-2} \), so
\[
x' = \int \sqrt{x^{-2}} \, dx = \int x^{-1} \, dx = \ln x + C
\]
We can take \( C = 0 \), and use
\[
x' = \ln x
\]
7. Using the notation of Lecture 37, let $S$ be the surface of revolution defined by
the functions $F(t) = t, G(t) = t, t \in (0, 2)$: $S$ is a part of a cone. It can be parameterized as

$$S = \{(x^2 \cos x^1, x^2 \sin x^1, x^2) \mid x^1 \in [0, 2\pi), x^2 \in (0, 2)\}.$$ 

We give $S$ the metric inherited from $\mathbb{R}^3$, and we denote by $\nabla$ the associated Riemannian connection on $S$.

Let $\gamma : [0, 2\pi] \rightarrow S$ be the circle at $x^2 = 1$ parameterized as

$$\gamma(\theta) = (\cos \theta, \sin \theta, 1)$$

(a) **(7 points)** Compute $\nabla_{\dot{\gamma}} \ddot{\gamma}$, the “covariant acceleration.”

**Solution:** Using the notation from Lecture 37, if we use coordinates $(x^1, x^2)$ on $S$, the Christoffel symbols are

$$\Gamma_{12}^1 = \Gamma_{21}^1 = F'(x^2)/F(x^2) = 1/x^2$$

$$\Gamma_{11}^1 = \Gamma_{22}^1 = 0$$

$$\Gamma_{11}^2 = \frac{-F(x^2)F'(x^2)}{F'(x^2)^2 + G'(x^2)^2} = \frac{-x^2}{2}$$

$$\Gamma_{22}^2 = \frac{F'(x^2)G''(x^2) + G'(x^2)F''(x^2)}{F'(x^2)^2 + G'(x^2)^2} = 0$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = 0$$

With respect to the coordinates $(x^1, x^2)$, the curve $\gamma$ is described as $x^1 = \theta$, $x^2 = 1$. So $\gamma$ has components $\gamma^1(\theta) = \theta$ and $\gamma^2(\theta) = 1$. Thus $\dot{\gamma}^1 = 1$ and $\dot{\gamma}^2 = 0$, and $\ddot{\gamma}^1 = \ddot{\gamma}^2 = 0$. The components of the covariant acceleration are

$$\nabla_{\dot{\gamma}} \ddot{\gamma}^1 = \ddot{\gamma}^1 + \sum_{i,j=1}^2 \Gamma_{ij}^1(\gamma) \dot{\gamma}^i \dot{\gamma}^j = \ddot{\gamma}^1 + \frac{2}{\gamma^2} \ddot{\gamma}^1 \gamma^2 = 0$$

$$\nabla_{\dot{\gamma}} \ddot{\gamma}^2 = \ddot{\gamma}^2 + \sum_{i,j=1}^2 \Gamma_{ij}^2(\gamma) \dot{\gamma}^i \dot{\gamma}^j = \ddot{\gamma}^2 + \frac{-\gamma^2}{2} \ddot{\gamma}^1 \gamma^1 = -1/2$$

In other words

$$\nabla_{\dot{\gamma}} \ddot{\gamma} = (-1/2) \frac{\partial}{\partial x^2}$$

(b) **(7 points)** Compute the parallel translation of the vector $\frac{\partial}{\partial x^1}|_{\gamma(0)} \in T_{\gamma(0)}S$ along the curve $\gamma$. (The answer is a vector in $T_{\gamma(2\pi)}S$.)
**Solution:** Let $v(\theta)$ denote a vector field along $\gamma$, so $v(\theta) \in T_{\gamma(\theta)}S$. We need to solve the initial value problem

$$\nabla_{\dot{\gamma}} v = 0, \quad v(0) = \frac{\partial}{\partial x^1}|_{\gamma(0)}$$

Letting the components of $v$ be $v^1$ and $v^2$, this becomes the system of differential equations

$$\dot{v}^1 + \sum_{i,j=1}^{2} \Gamma^1_{ij}(\gamma) \dot{\gamma}^i v^j = 0$$

$$\dot{v}^2 + \sum_{i,j=1}^{2} \Gamma^2_{ij}(\gamma) \dot{\gamma}^i v^j = 0$$

with initial conditions

$$v^1(0) = 1, \quad v^2(0) = 0.$$ 

Using the Christoffel symbols $\Gamma^1_{12}(\gamma) = \Gamma^1_{21}(\gamma) = 1/\gamma^2 = 1$ and $\Gamma^2_{11}(\gamma) = -\gamma^2/2 = -1/2$ as well as $\dot{\gamma}^1 = 1$, $\dot{\gamma}^2 = 0$, this simplifies to

$$\dot{v}^1 + v^2 = 0$$

$$\dot{v}^2 + (-1/2) v^1 = 0$$

To solve, take the derivative of the first equation to get $\ddot{v}^1 + \dot{v}^2 = 0$, and substitute in for $\dot{v}^2$ using the second equation, so that we have

$$\ddot{v}^1 + (1/2) v^1 = 0$$

The general solution is then

$$v^1(\theta) = A \cos(\theta/\sqrt{2}) + B \sin(\theta/\sqrt{2})$$

Using the initial condition $v^1(0) = 1$, we find $A = 1$. Then

$$v^2 = -\dot{v}^1 = (1/\sqrt{2}) \sin(\theta/\sqrt{2}) - (B/\sqrt{2}) \cos(\theta/\sqrt{2})$$

The initial condition $v^2(0) = 0$ then means $B = 0$. Thus the solution is

$$v^1(\theta) = \cos(\theta/\sqrt{2}), \quad v^2(\theta) = (1/\sqrt{2}) \sin(\theta/\sqrt{2})$$

When we parallel translate all around $\gamma$, we end at $\theta = 2\pi$, so we take

$$v^1(2\pi) = \cos(2\pi/\sqrt{2}), \quad v^2(2\pi) = (1/\sqrt{2}) \sin(2\pi/\sqrt{2})$$

Thus the parallel translated vector is

$$\cos\left(\pi \sqrt{2}\right) \frac{\partial}{\partial x^1}|_{\gamma(2\pi)} + \frac{\sin(\pi \sqrt{2})}{\sqrt{2}} \frac{\partial}{\partial x^2}|_{\gamma(2\pi)}$$