Instructions:

- Write your name at the top of this page.
- There are 41 points possible on this exam. Take note that the problems are not weighted equally.
- For problems that ask you to prove something, you are allowed to use in your proof any result from the lectures or the relevant sections of the textbook.
- No books, notes, calculators, or other aids are permitted.
- The last page is for work that does not fit on other pages.

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1. In class we introduced the two-dimensional sphere

\[ S^2 = \left\{ x \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \right\}, \]

and the two-dimensional real projective space

\[ \mathbb{RP}^2 = \text{the set of lines through the origin in } \mathbb{R}^3. \]

There is a map \( F : S^2 \rightarrow \mathbb{RP}^2 \) defined as \( F(x^1, x^2, x^3) = [x^1, x^2, x^3] \). (As usual the square brackets denote “the line spanned by” the given vector.)

(a) (5 points) Prove that \( F : S^2 \rightarrow \mathbb{RP}^2 \) is a smooth map. [Strictly speaking, this requires checking something at every point in the domain. For this exam, it will suffice if you check smoothness at any one point in \( S^2 \) of your choosing.]

Pick \( p = (0, 0, 1) \in S^2 \), \( F(p) = [0, 0, 1] \in \mathbb{RP}^2 \).

Chart on \( S^2 \):
\[ \mathcal{U} = \{ x \in S^2 \mid x^3 > 0 \} \]
\[ \phi(x^1, x^2, x^3) = (x^1, x^2) \]
\[ \phi^{-1}(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \]
on \( \phi(U) = \{ (u, v) \mid u^2 + v^2 < 1 \} \subset \mathbb{R}^2 \)

Chart on \( \mathbb{RP}^2 \):
\[ \mathcal{V} = \{ [x^1, x^2, x^3] \mid x^3 \neq 0 \} \]
\[ \psi([x^1, x^2, x^3]) = \left( \frac{x^1}{x^3}, \frac{x^2}{x^3} \right) \]

Look at \( \psi \circ F \circ \phi^{-1} \):
\[ \psi(F(\phi^{-1}(u, v))) = \psi\left(F(u, v, \sqrt{1 - u^2 - v^2})\right) = \psi([u, v, \sqrt{1 - u^2 - v^2}]) \]
\[ = \left( \frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}} \right) \]

Since \( u^2 + v^2 < 1 \), the expression under square root is always positive, so \( \sqrt{1 - u^2 - v^2} \) is smooth and never zero.

Since ratios of smooth functions are smooth when the denominator is not zero, we see \( \frac{u}{\sqrt{1 - u^2 - v^2}} \) and \( \frac{v}{\sqrt{1 - u^2 - v^2}} \) are smooth.
(b) (3 points) What are the critical points of $F : S^2 \to \mathbb{R}P^2$? Work is not required for this problem, just state the answer.

There are no critical points.

(c) (2 points) True or False: The map $F : S^2 \to \mathbb{R}P^2$ is a diffeomorphism. Circle your answer: True False

(d) (2 points) Let $U \subset S^2$ denote the upper hemisphere:

$$U = \{ x \in S^2 \mid x^3 > 0 \}$$

And let $F(U) \subset \mathbb{R}P^2$ be its image under $F$. True or False: $U$ is diffeomorphic to $F(U)$. Circle your answer: True False

(e) (1 point) True or False: The one-dimensional sphere $S^1$ and the one-dimensional real projective space $\mathbb{R}P^1$ are diffeomorphic. Circle your answer: True False
2. Consider \( \mathbb{R}^2 \) with standard coordinates \((x^1, x^2)\). Define a non-standard coordinate system \((y^1, y^2)\) by the relations
\[
\begin{align*}
y^1 &= x^1 \\
y^2 &= x^1 + x^2
\end{align*}
\]
We can think of this as defining a non-standard chart \((\mathbb{R}^2, \varphi)\) where \( \varphi(x^1, x^2) = (x^1, x^1 + x^2) \). Let \( p \) be the point whose standard coordinates are \( x^1 = 1 \), \( x^2 = \pi \). Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be the function \( f(x^1, x^2) = \sin(x^2) \).

(a) (3 points) Prove that the non-standard chart is compatible with the standard coordinate chart.

\[
\text{Standard chart is } (\mathbb{R}^2, \text{Id}) \quad \text{Id}(x^1, x^2) = (x^1, x^2)
\]
Intersection of domains: \( \mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2 \) which is open.

Coordinate change \( \varphi \circ \text{Id}^{-1}(x^1, x^2) = \varphi(x^1, x^2) = (x^1, x^1 + x^2) \)
which is linear, hence smooth.

OR:

Inverse coord change \( \text{Id} \circ \varphi^{-1}(y^1, y^2) = (y^1, y^2 - y^1) \)
which is linear, hence smooth.

(b) (5 points) Compute \( \frac{\partial f}{\partial y^1} \bigg|_p \) and \( \frac{\partial f}{\partial y^2} \bigg|_p \).

Write \( f \) in terms of \((y^1, y^2)\):

\[
\begin{align*}
f \circ \varphi^{-1}(y^1, y^2) &= f(y^1, y^2 - y^1) = \sin(y^2 - y^1) \\
\frac{\partial}{\partial y^1} \sin(y^2 - y^1) &= -\cos(y^2 - y^1) \\
\frac{\partial}{\partial y^2} \sin(y^2 - y^1) &= \cos(y^2 - y^1)
\end{align*}
\]
At \( p \): \( y^1 = 1 \), \( y^2 = \pi + 1 \) so \( y^2 - y^1 = \pi \)

\[
\begin{align*}
\frac{\partial f}{\partial y^1} \bigg|_p &= -\cos(\pi) = 1 \\
\frac{\partial f}{\partial y^2} \bigg|_p &= \cos(\pi) = -1
\end{align*}
\]
(c) (4 points) Write a system of linear equations that relates the two bases for $T_p\mathbb{R}^2$:

$$\left\{ \frac{\partial}{\partial x^1} \big|_p, \frac{\partial}{\partial x^2} \big|_p \right\} \text{ and } \left\{ \frac{\partial}{\partial y^1} \big|_p, \frac{\partial}{\partial y^2} \big|_p \right\} :$$

\[
\begin{align*}
\frac{\partial}{\partial x^1} &= \frac{\partial y^1}{\partial x^1} \frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^1} \frac{\partial}{\partial y^2} = 1 \frac{\partial}{\partial y^1} + 1 \frac{\partial}{\partial y^2} \\
\frac{\partial}{\partial x^2} &= \frac{\partial y^1}{\partial x^2} \frac{\partial}{\partial y^1} + \frac{\partial y^2}{\partial x^2} \frac{\partial}{\partial y^2} = 0 \frac{\partial}{\partial y^1} + 1 \frac{\partial}{\partial y^2}
\end{align*}
\]

So

\[
\begin{align*}
\frac{\partial}{\partial x^1} \big|_p &= \frac{\partial}{\partial y^1} \big|_p + \frac{\partial}{\partial y^2} \big|_p \\
\frac{\partial}{\partial x^2} \big|_p &= \frac{\partial}{\partial y^2} \big|_p
\end{align*}
\]

(d) (3 points) Regard $f : \mathbb{R}^2 \to \mathbb{R}$ as a smooth map. Using the basis $\left\{ \frac{\partial}{\partial y^1} \big|_p, \frac{\partial}{\partial y^2} \big|_p \right\}$ for $T_p\mathbb{R}^2$, and using a basis for $T_{f(p)}\mathbb{R}$ of your choosing, write a matrix representative for $f_* : T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$.

Let $x$ be standard coordinate on $\mathbb{R}$, $f(p) = \sin(\pi x) = 0$, and $\frac{\partial}{\partial x} \big|_0$ is a basis for $T_0\mathbb{R}$.

Thus

\[
\begin{align*}
f_* \left( \frac{\partial}{\partial y^1} \big|_p \right) &= \frac{\partial f}{\partial y^1} \big|_p \frac{\partial}{\partial x} \big|_0 = 1 \frac{\partial}{\partial x} \big|_0 \\
f_* \left( \frac{\partial}{\partial y^2} \big|_p \right) &= \frac{\partial f}{\partial y^2} \big|_p \frac{\partial}{\partial x} \big|_0 = -1 \frac{\partial}{\partial x} \big|_0
\end{align*}
\]

So \([f_*] = \begin{bmatrix} 1 & -1 \end{bmatrix}\)
3. Let $g : \mathbb{R}^3 \to \mathbb{R}$ and $h : \mathbb{R}^3 \to \mathbb{R}$ be two smooth functions on $\mathbb{R}^3$, and consider the subset 

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = 0 \text{ and } h(x, y, z) = 0\}.$$ 

Assume that, for any $p \in C$, the gradients $\nabla g(p)$ and $\nabla h(p)$ have non-zero cross product:

$$\nabla g(p) \times \nabla h(p) \neq 0.$$ 

(a) (6 points) Prove that $C$ is a submanifold of $\mathbb{R}^3$.

Define a map $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $F(x, y, z) = (g(x, y, z), h(x, y, z))$.

Then $C = F^{-1}(0, 0)$. Using standard bases,

$$\left[F^*\right] = \begin{bmatrix}
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\
\frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}
\end{bmatrix} : T_p \mathbb{R}^3 \to T_{F(p)} \mathbb{R}^2$$

We know that at each $p \in C$, $\nabla g(p) \times \nabla h(p) \neq 0$.

This implies that $\nabla g(p)$ and $\nabla h(p)$ are linearly independent.

$\Rightarrow$ the rows of $\left[F^*\right]$ are linearly independent.

$\Rightarrow \text{rank} \left[F^*\right] = 2 \Rightarrow p \in C$ is not a critical point.

Since no points of $C = F^{-1}(0, 0)$ is a critical point, $(0, 0)$ is a regular value of $F$, and by the regular value theorem, $C$ is a submanifold of $\mathbb{R}^3$.

(b) (2 points) What is the dimension of $C$ (as a manifold in its own right)? 
Circle your answer: 0 1 2 3

6
4. (5 points) Let \( M = \{ x \in \mathbb{R} \mid x > 0 \} \) be the set of positive real numbers. It is an open subset of \( \mathbb{R} \), hence a manifold of dimension 1. Define a vector field \( V \) on \( M \) by the formula
\[
V(x) = 2x \frac{\partial}{\partial x}
\]
Solve for the corresponding one-parameter group of diffeomorphisms of \( M \). Write your answer as a map \( \varphi_t : M \to M \), and show work that justifies your answer.

Initial value problem: \( \gamma : \mathbb{R} \to M \)
\[
\gamma_x \left( \frac{\partial}{\partial t} \right) = \frac{dx}{dt} \frac{\partial}{\partial x} = V(\gamma(t)) = 2\gamma(t) \frac{\partial}{\partial x}
\]
Thus \( \frac{dx}{dt} = 2x \Rightarrow \gamma(t) = Ce^{2t} \)
Initial condition \( \gamma(0) = x_0 \) , we take \( C = x_0 \)
so \( \gamma_{x_0}(t) = x_0 e^{2t} \) is the particular solution.
The map is \( \varphi_t(x) = \gamma_x(t) = xe^{2t} \).
Use this page for work that does not fit on other pages.