Math 481: Midterm 2 Solutions

Name:

Wednesday, April 15, 2020

Instructions:

• There are 49 points possible on this exam. Take note that the problems are not weighted equally.

• Please complete the problems and upload your solutions to Box, just as you would do with a homework assignment.

• For problems that ask you to prove something, you are allowed to use in your proof any result from the lectures or the relevant sections of the textbook.

• For this take-home exam you may refer to your notes, the lecture videos, homework solutions, and any of resources the mentioned on the course website. You may not refer to online sources not mentioned on the course website (e.g. google searches), and you may not discuss the exam with other students.

• If you have questions (for example, needing clarification of a problem statement). Please email the instructor at jpascale@illinois.edu.

• This exam will be posted online no later than 7:00am CDT on Wednesday, April 15, and your solutions should be uploaded by 3:00am CDT on Saturday, April 18.

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In the problems that follow, $M$ refers to the surface of an ellipsoid:

$$M = \left\{(x, y, z) \in \mathbb{R}^3 \left| \frac{x^2}{2} + y^2 + \frac{z^2}{3} = 1 \right. \right\},$$

and $\gamma : [0, 2\pi] \times [0, \pi] \rightarrow M$ is a parameterization of $M$ by

$$\gamma(u, v) = (\sqrt{2}\cos(u)\sin(v), \sin(u)\sin(v), \sqrt{3}\cos(v)), \quad (u, v) \in [0, 2\pi] \times [0, \pi].$$

This parameterization is one-to-one on the interior $(0, 2\pi) \times (0, \pi)$ of the domain rectangle, and so its inverse defines a coordinate system on $U = \gamma((0, 2\pi) \times (0, \pi))$.

Let $i : M \rightarrow \mathbb{R}^3$ be the inclusion map $i(x, y, z) = (x, y, z)$.

1. (12 points) We define a Riemannian metric $g$ on $M$ by declaring that, for a given point $p \in M$, and given tangent vectors $X, Y \in T_pM$, we have

$$g(p)(X, Y) = (i_*(X)) \cdot (i_*(Y))$$

where $i_* : T_pM \rightarrow T_p\mathbb{R}^3$ is the derivative of $i$, and $(i_*(X)) \cdot (i_*(Y))$ denotes the ordinary dot product of the vectors $i_*(X), i_*(Y) \in T_p\mathbb{R}^3 \cong \mathbb{R}^3$.

Write a formula for $g$ in terms of the $(u, v)$ coordinate system on $U \subset M$. That is, find the four coefficient functions $g_{11}, g_{12}, g_{21}, g_{22}$ such that

$$g = g_{11} \, du \otimes du + g_{12} \, du \otimes dv + g_{21} \, dv \otimes du + g_{22} \, dv \otimes dv$$

where each $g_{ij}$ is a function of $u$ and $v$. **Hint:** In the correct answer, each coefficient $g_{ij}$ is combination of functions like $\sin(u), \cos(u), \sin(v), \cos(v)$. You do not need to simplify your answer.

**Solution:** The first step is to work out the matrix of $i_*$ with respect to appropriate bases. As a basis for $T_pM$, we use the tangent vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ coming from the $(u, v)$ coordinates on $U$. As a basis for $T_p\mathbb{R}^3$, we use the standard basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. The mapping from the $(u, v)$ coordinates on $U$ to the $(x, y, z)$ coordinates on $\mathbb{R}^3$ is $i \circ \gamma$, and the formula for this is the same as that for $\gamma$. Thus the matrix of $i_*$ with respect to these bases is nothing but the matrix of partial derivatives of the formula for $\gamma$:

$$[i_*] = \begin{bmatrix} -\sqrt{2}\sin(u)\sin(v) & 2\cos(u)\cos(v) \\ \cos(u)\sin(v) & \sin(u)\cos(v) \\ 0 & -\sqrt{3}\sin(v) \end{bmatrix}$$

Or,

$$i_* \left( \frac{\partial}{\partial u} \right) = -\sqrt{2}\sin(u)\sin(v)\frac{\partial}{\partial x} + \cos(u)\sin(v)\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}$$

$$i_* \left( \frac{\partial}{\partial v} \right) = \sqrt{2}\cos(u)\cos(v)\frac{\partial}{\partial x} + \sin(u)\cos(v)\frac{\partial}{\partial y} + -\sqrt{3}\sin(v)\frac{\partial}{\partial z}$$

We need to compute

$$g_{11} = g \left( i_* \left( \frac{\partial}{\partial u} \right), i_* \left( \frac{\partial}{\partial u} \right) \right) = \left( i_* \left( \frac{\partial}{\partial u} \right) \right) \cdot \left( i_* \left( \frac{\partial}{\partial u} \right) \right)$$

$$g_{12} = g_{21} = g \left( i_* \left( \frac{\partial}{\partial u} \right), i_* \left( \frac{\partial}{\partial v} \right) \right) = \left( i_* \left( \frac{\partial}{\partial u} \right) \right) \cdot \left( i_* \left( \frac{\partial}{\partial v} \right) \right)$$

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\[ g_{22} = g \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = \left( i_* \left( \frac{\partial}{\partial v} \right) \right) \cdot \left( i_* \left( \frac{\partial}{\partial v} \right) \right) \]

Since the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \) is orthonormal for the standard dot product, we compute

\[
g_{11} = [-\sqrt{2} \sin(u) \sin(v)]^2 + [\cos(u) \sin(v)]^2 + b^2 = 2 \sin^2(u) \sin^2(v) + \cos^2(u) \sin^2(v)
\]

\[
= (2 \sin^2(u) + \cos^2(u)) \sin^2(v) = (\sin^2(u) + 1) \sin^2(v)
\]

\[
g_{12} = g_{21} = [-\sqrt{2} \sin(u) \sin(v)][\sqrt{2} \cos(u) \cos(v)] + [\cos(u) \sin(v)][\sin(u) \cos(v)] + [0][\sqrt{3} \sin(v)]
\]

\[
= -2 \sin(u) \cos(u) \sin(v) \cos(v) + \sin(u) \cos(u) \sin(v) \cos(v)
\]

\[
= -\sin(u) \cos(u) \sin(v) \cos(v)
\]

\[
g_{22} = [\sqrt{2} \cos(u) \cos(v)]^2 + [\sin(u) \cos(v)]^2 + [-\sqrt{3} \sin(v)]^2
\]

\[
= 2 \cos^2(u) \cos^2(v) + \sin^2(u) \cos^2(v) + 3 \sin^2(v)
\]

2. **(6 points)** Set up an integral to compute length of the curve \( C \) on \( M \) parameterized by \( \gamma(t, \pi/2), \ t \in [0, 2\pi] \). Your answer should be a standard single-variable integral with an explicit integrand.

**Solution:** In the \( (u, v) \) coordinate system, the parametric equations for \( C \) are \( u = t, \ v = \pi/2 \). Thus the velocity vector of \( C \) is simply \( \frac{\partial}{\partial u} \), and the speed is

\[
\sqrt{g(\frac{\partial}{\partial u}, \frac{\partial}{\partial u})} = \sqrt{g_{11}},
\]

where \( g_{11} \) is evaluated at the point whose coordinates are \( u = t, v = \pi/2 \). Using the result of the previous part, we have

\[
g_{11}|_{u=t,v=\pi/2} = (\sin^2(t) + 1) \sin^2(\pi/2) = \sin^2(t) + 1
\]

Thus the length is

\[
L(C) = \int_0^{2\pi} \sqrt{\sin^2(t) + 1} \ dt
\]

**Alternative solution:** One can observe that, via the inclusion \( i : M \to \mathbb{R}^3 \), the curve \( C \) is also a curve in \( \mathbb{R}^3 \), and because the metric on \( M \) is defined in terms of the standard metric on \( \mathbb{R}^3 \), the length of \( C \) is the same whether it is computed in \( M \) or in \( \mathbb{R}^3 \). Then we observe that \( C \) is an ellipse in the \( x y \)-plane parameterized by

\[
\eta(t) = (\sqrt{2} \cos(t), \sin(t), 0).
\]

The velocity vector is

\[
\hat{\eta}(t) = (-\sqrt{2} \sin(t), \cos(t), 0)
\]

and the speed is

\[
\sqrt{\eta \cdot \hat{\eta}} = \sqrt{2 \sin^2(t) + \cos^2(t)} = \sqrt{\sin^2(t) + 1},
\]

so we get the same answer as above.

3. **(3 points each)** Consider the 1-form on \( \mathbb{R}^3 \) given by \( \alpha = dz - y \, dx \).


(a) Compute $d\alpha$;

Solution:

$$d\alpha = d(dx - ydy) = d(1) \wedge dz - d(y) \wedge dx = 0 \wedge dz - dy \wedge dx = dx \wedge dy$$

(b) Compute $\alpha \wedge d\alpha$;

Solution:

$$\alpha \wedge d\alpha = (dz - ydy) \wedge (dx \wedge dy) = dz \wedge dx \wedge dy - ydx \wedge dx \wedge dy$$

$$= dz \wedge dx \wedge dy - 0 = dx \wedge dy \wedge dz$$

(c) Compute $i^*\alpha$ in the $(u, v)$ coordinate system;

Solution: First we compute the action of $i^*$ on 1-forms. Using the basis $dx, dy, dz$ for $T_p\mathbb{R}^3$, and the basis $du, dv$ for $T_p^*M$, the matrix of $i^*$ is just the transpose of the matrix for $i_*$ found in the first problem. Thus

$$i^*(dx) = -\sqrt{2}\sin(u)\sin(v)\, du + \sqrt{2}\cos(u)\cos(v)\, dv$$

$$i^*(dy) = \cos(u)\sin(v)\, du + \sin(u)\cos(v)\, dv$$

$$i^*(dz) = 0\, du - \sqrt{3}\sin(v)\, dv$$

Now we compute using the relation $y = \sin(u)\sin(v)$

$$i^*\alpha = i^*(dz - ydx) = i^*(dz) - \sin(u)\sin(v)i^*(dx)$$

$$= -\sqrt{3}\sin(v)\, dv - \sin(u)\sin(v)[-\sqrt{2}\sin(u)\sin(v)\, du + \sqrt{2}\cos(u)\cos(v)\, dv]$$

$$= \sqrt{2}\sin^2(u)\sin^2(v)\, du - [\sqrt{3}\sin(v) + \sqrt{2}\sin(u)\sin(v)\cos(u)\cos(v)]\, dv$$

(d) Compute $d(i^*\alpha)$ in the $(u, v)$ coordinate system;

Solution:

$$d(i^*\alpha) = i^*d\alpha = i^*(dx \wedge dy) = i^*(dx) \wedge i^*(dy)$$

$$= (-\sqrt{2}\sin(u)\sin(v)\, du + \sqrt{2}\cos(u)\cos(v)\, dv) \wedge (\cos(u)\sin(v)\, du + \sin(u)\cos(v)\, dv)$$

$$= -\sqrt{2}\sin(u)\sin(v)\sin(u)\cos(v)\, du \wedge dv + \sqrt{2}\cos(u)\cos(v)\cos(u)\sin(v)\, dv \wedge du$$

$$= -\sqrt{2}\sin(u)\sin(v)\sin(u)\cos(v)\, du \wedge dv - \sqrt{2}\cos(u)\cos(v)\cos(u)\sin(v)\, du \wedge dv$$

$$= -\sqrt{2}[\sin(u)\sin(v)\sin(u)\cos(v) + \cos(u)\cos(v)\cos(u)\sin(v)]\, du \wedge dv$$

$$= -\sqrt{2}\sin(v)\cos(v)[\sin^2(u) + \cos^2(u)]\, du \wedge dv$$

$$= -\sqrt{2}\sin(v)\cos(v)\, du \wedge dv$$

(e) Compute $i^*(\alpha \wedge d\alpha)$ in the $(u, v)$ coordinate system.

Solution: Because $i^*(\alpha \wedge d\alpha)$ is a three-form, and $M$ is a two-dimensional manifold, and $3 > 2$, this form must be zero.
4. (6 points) Let $\alpha$ be as in the previous problem, and let $N = M \cap \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$ be the upper half of the ellipsoid. Compute $\int_N d(i^* \alpha)$; use the orientation on $M$ that makes $\gamma$ a positive parameterization.

**Solution:** In terms of the parameterization $\gamma$, the subset $N$ corresponds to the subset of the domain where $0 \leq u \leq \pi/2$. Thus we must compute the integral $\int_{[0,2\pi] \times [0,\pi/2]} \gamma^*(d(i^* \alpha))$. Because $\gamma$ is used to define the $(u, v)$ coordinate system, we know that $\gamma^*(d(i^* \alpha))$ is nothing but the expression of $d(i^* \alpha)$ in the $(u, v)$ coordinate system, which is $-\sqrt{2}\sin(v)\cos(v) du \wedge dv$. Because we are assuming that $\gamma$ is a positive parameterization, we don’t have to rearrange the $du$ and $dv$ in this expression. Thus

\[
\int_N d(i^* \alpha) = \int_{[0,2\pi] \times [0,\pi/2]} -\sqrt{2}\sin(v)\cos(v) du \wedge dv = \int_0^\pi \int_0^{\pi/2} -\sqrt{2}\sin(v)\cos(v) du dv
\]

Now

\[
\int_0^{\pi/2} \sin(v)\cos(v) dv = \int_0^{\pi/2} \frac{1}{2} \sin(2v) dv = \left[ -\frac{1}{4} \cos(2v) \right]_{v=0}^{v=\pi/2} = -\frac{1}{4} [\cos(\pi) - \cos(0)] = -\frac{1}{4} [-1 - 1] = \frac{1}{2}
\]

Thus

\[
\int_N d(i^* \alpha) = -\pi \sqrt{2}.
\]

5. Let $V$ be a vector space. Given a vector $v \in V$, and an alternating tensor $\beta \in \Lambda^k(V)$, we can form an alternating tensor $i_v \beta \in \Lambda^{k-1}(V)$, defined by the rule

\[(i_v \beta)(v_1, v_2, \ldots, v_{k-1}) = \beta(v, v_1, v_2, \ldots, v_{k-1})\]

[In other words, $i_v \beta$ is like $\beta$, but the first input is always the fixed vector $v$, so $i_v \beta$ only depends on $k - 1$ inputs.]

(a) (5 points) Show that if $v, w \in V$, and $\beta \in \Lambda^k(V)$, then

\[i_v(i_w \beta) = -i_w(i_v \beta),\]

where each side of the equation is an element of $\Lambda^{k-2}(V)$.

**Solution:** Evaluate each side on a sequence of $k - 1$ vectors $v_1, \ldots, v_{k-2}$:

\[(i_v(i_w \beta))(v_1, \ldots, v_{k-2}) = (i_w \beta)(v, v_1, \ldots, v_{k-2}) = \beta(w, v, v_1, \ldots, v_{k-2}) = -\beta(w, v_1, v_2, \ldots, v_{k-2}) = -(i_w \beta)(v_1, \ldots, v_{k-2}) = -(i_w(i_v \beta))(v_1, \ldots, v_{k-2})\]

Since $i_v(i_w \beta)$ and $-i_w(i_v \beta)$ are equal on any set of inputs, they are the same tensor.

(b) (5 points) Assume $n = 3$ and let $e_1, e_2, e_3$ be a basis of $V$, and let $\sigma^1, \sigma^2, \sigma^3$ be the dual basis for $V^*$. Compute

\[i_{e_2}(\sigma^1 \wedge \sigma^2 + \sigma^2 \wedge \sigma^3)\]

The result is an element of $\Lambda^1(V) = V^*$; Write your answer in terms of the basis $\sigma^1, \sigma^2, \sigma^3$. 

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Solution: we evaluate this tensor on an arbitrary vector \( v \).

\[
(i_{e_2}(\sigma^1 \wedge \sigma^2 + \sigma^2 \wedge \sigma^3))(v) = (\sigma^1 \wedge \sigma^2 + \sigma^2 \wedge \sigma^3)(e_2, v)
\]

\[
= (\sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1 + \sigma^2 \otimes \sigma^3 - \sigma^3 \otimes \sigma^2)(e_2, v)
\]

\[
= \sigma^1(e_2)\sigma^2(v) - \sigma^2(e_2)\sigma^1(v) + \sigma^2(e_2)\sigma^3(v) - \sigma^3(e_2)\sigma^2(v)
\]

\[
= 0 \cdot \sigma^2(v) - 1 \cdot \sigma^1(v) + 1 \cdot \sigma^3(v) - 0 \cdot \sigma^2(v)
\]

\[
= -\sigma^1(v) + \sigma^3(v) = (-\sigma^1 + \sigma^3)(v)
\]

Since this is true for any input \( v \), we conclude that

\[
i_{e_2}(\sigma^1 \wedge \sigma^2 + \sigma^2 \wedge \sigma^3) = -\sigma^1 + \sigma^3
\]