Math 481: Homework 2 Solutions
Due Wednesday, February 12, 2020

1. (a) Propose a smooth atlas for the set $S^1 \times S^1$.

Solution: [Remark: There are many atlases for $S^1$, here we will use the one from the lecture, which is not necessarily the easiest one.] Let us start with the atlas on $S^1$ described in the lecture. It has four charts:

$$S^1 = \{(x^1, x^2) \in \mathbb{R}^2 \mid (x^1)^2 + (x^2)^2 = 1\}$$

described in the lecture. It has four charts:

$$U^+_1 = \{x \in S^1 \mid x^1 > 0\}, \quad \phi^+_1(x) = x^2;$$
$$U^-_1 = \{x \in S^1 \mid x^1 < 0\}, \quad \phi^-_1(x) = x^2;$$
$$U^+_2 = \{x \in S^1 \mid x^2 > 0\}, \quad \phi^+_2(x) = x^1;$$
$$U^-_2 = \{x \in S^1 \mid x^2 < 0\}, \quad \phi^-_2(x) = x^1.$$ 

The set $S^1 \times S^1$ is the set of pairs of points in $S^1$. We can think of this as a subset of $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, which we can write as

$$S^1 \times S^1 = \{(x^1, x^2, y^1, y^2) \in \mathbb{R}^4 \mid (x^1)^2 + (x^2)^2 = 1, \quad (y^1)^2 + (y^2)^2 = 1\}$$

(We use $(x^1, x^2)$ as variables on the first $S^1$, and $(y^1, y^2)$ on the second $S^1$.) The idea for the atlas on $S^1 \times S^1$ is that the charts on the product are products of the charts on $S^1$. Thus there will be $4 \times 4 = 16$ total charts in our atlas. The naming scheme will be $U^i_{ij}$, where $i$ and $j$ can be either 1 or 2, and $s$ and $t$ can be either + or −. We define $U^i_{ij}$ to be the subset of $S^1 \times S^1$ where $sx^i > 0$ and $ty^j > 0$. The coordinate map $\phi^i_{ij}$ is then always obtained by “taking the other coordinates.” Explicitly, here are the sixteen charts:

$$U^{++}_{11} = \{(x, y) \in S^1 \times S^1 \mid x^1 > 0, y^1 > 0\}, \quad \phi^{++}_{11}(x, y) = (x^2, y^2);$$
$$U^{+-}_{11} = \{(x, y) \in S^1 \times S^1 \mid x^1 < 0, y^1 > 0\}, \quad \phi^{+-}_{11}(x, y) = (x^2, y^2);$$
$$U^{-+}_{21} = \{(x, y) \in S^1 \times S^1 \mid x^2 > 0, y^1 > 0\}, \quad \phi^{-+}_{21}(x, y) = (x^1, y^2);$$
$$U^{- -}_{21} = \{(x, y) \in S^1 \times S^1 \mid x^2 < 0, y^1 > 0\}, \quad \phi^{- -}_{21}(x, y) = (x^1, y^2);$$

$$U^{-+}_{11} = \{(x, y) \in S^1 \times S^1 \mid x^1 > 0, y^1 < 0\}, \quad \phi^{-+}_{11}(x, y) = (x^2, y^2);$$
$$U^{+-}_{11} = \{(x, y) \in S^1 \times S^1 \mid x^1 < 0, y^1 < 0\}, \quad \phi^{+-}_{11}(x, y) = (x^2, y^2);$$
$$U^{- -}_{21} = \{(x, y) \in S^1 \times S^1 \mid x^2 > 0, y^1 < 0\}, \quad \phi^{- -}_{21}(x, y) = (x^1, y^2);$$
$$U^{--}_{21} = \{(x, y) \in S^1 \times S^1 \mid x^2 < 0, y^1 < 0\}, \quad \phi^{--}_{21}(x, y) = (x^1, y^2);$$

$$U^{++}_{12} = \{(x, y) \in S^1 \times S^1 \mid x^1 > 0, y^2 > 0\}, \quad \phi^{++}_{12}(x, y) = (x^2, y^1);$$
$$U^{+-}_{12} = \{(x, y) \in S^1 \times S^1 \mid x^1 < 0, y^2 > 0\}, \quad \phi^{+-}_{12}(x, y) = (x^2, y^1);$$
$$U^{-+}_{22} = \{(x, y) \in S^1 \times S^1 \mid x^2 > 0, y^2 > 0\}, \quad \phi^{-+}_{22}(x, y) = (x^1, y^1);$$

$$U^{--}_{22} = \{(x, y) \in S^1 \times S^1 \mid x^2 < 0, y^2 > 0\}, \quad \phi^{--}_{22}(x, y) = (x^1, y^1);$$
(b) Verify that one of the charts of your atlas satisfies the definition of a chart.

**Solution:** Let us consider the chart \((U_{11}^+, \phi_{11}^+))\). We need to check that \(\phi_{11}^+(U_{11}^+)\) is open and \(\phi_{11}^+\) is one-to-one. In \(U_{11}^+\), the coordinates \(x^1\) and \(y^1\) are positive, and from the equations \((x^1)^2 + (x^2)^2 = 1\) and \((y^1)^2 + (y^2)^2 = 1\), we see that \(x^2\) and \(y^2\) can take any values between \(-1\) and \(1\), not including the endpoints. Thus

\[
\phi_{11}^+(U_{11}^+) = \{(u, v) \in \mathbb{R}^2 \mid -1 < u < 1, \ -1 < v < 1\}
\]

which is nothing but a square of side length \(2\) with the boundary removed. To see that this set is open, just observe that, at any point \((u, v)\) in this square, the ball of radius \(r = \min(1 - |u|, 1 - |v|)\) fits inside the square.

To see that \(\phi_{11}^+\) is one-to-one, we can try to find the inverse function from \(\{(u, v) \in \mathbb{R}^2 \mid -1 < u < 1, \ -1 < v < 1\}\) to \(U_{11}^+\). Suppose that \((x^1, x^2, y^1, y^2) \in U_{11}^+\) and \(\phi_{11}^+(x, y) = (u, v)\). Then we know that \(x^2 = u, y^2 = v, x^1 > 0,\) and \(y^1 > 0\). From the equations \((x^1)^2 + (x^2)^2 = 1\) and \((y^1)^2 + (y^2)^2 = 1\), we see that \((x^1)^2 = 1 - u^2\) and \((y^1)^2 = 1 - v^2\), which together with positivity of \(x^1\) and \(y^1\) means \(x^1 = \sqrt{1 - u^2}\) and \(y^1 = \sqrt{1 - v^2}\). This argument shows that \(u\) and \(v\) uniquely determine a point in \(U_{11}^+\), so \(\phi_{11}^+\) is one-to-one. The inverse function is

\[
(\phi_{11}^+)^{-1}(u, v) = \left(\sqrt{1 - u^2}, u, \sqrt{1 - v^2}, v\right).
\]

(c) Verify that two of your charts with intersecting domains are compatible.

**Solution:** Let’s consider the pair \((U_{11}^+, \phi_{11}^+)\) and \((U_{22}^+, \phi_{22}^+)\). The intersection of the domains \(U_{11}^+ \cap U_{22}^+\) is the subset of \(S^1 \times S^1\) where \(x^1, x^2, y^1, y^2\) are all positive. The image of this set under \(\phi_{11}^+\) is the set of pairs \((u, v)\) where \(0 < u < 1\) and \(0 < v < 1\). This is again a square with the boundary removed, and it is open: the ball of radius

\[
r = \min(u, 1 - u, v, 1 - v)
\]

around the point \((u, v)\) always fits inside the square.

The transition function is \(\phi_{22}^+ \circ (\phi_{11}^+)^{-1}\) defined on the domain \(\phi_{11}^+(U_{11}^+ \cap U_{22}^+)\), and we must show that it is smooth. From the formula for \((\phi_{11}^+)^{-1}\) found in the previous part, we have

\[
\phi_{22}^+((\phi_{11}^+)^{-1}(u, v)) = \phi_{22}^+\left(\sqrt{1 - u^2}, u, \sqrt{1 - v^2}, v\right) = \left(\sqrt{1 - u^2}, \sqrt{1 - v^2}\right)
\]

This function is indeed smooth on the domain \(\{(u, v) \mid 0 < u < 1, \ 0 < v < 1\}\): the quantities under the square root signs are polynomials in the coordinates, and the defining inequalities of the domain imply that these quantities are always positive, so taking the square root is a smooth operation.
2. Use the following hints to prove that for $U \subset \mathbb{R}^n$ open, the map $\mathcal{C} : \mathbb{R}^n \to T_xU$, $v \mapsto D_v$ is a bijection, where

$$D_v(f) = \left. \frac{d}{dt} \right|_{t=0} f(x + tv) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)v^i.$$  

(a) Show that $\mathcal{C}$ is a linear map.

**Solution:** Let $v, w$ be vectors in $\mathbb{R}^n_x$, and let $f$ be any smooth function. Then

$$D_{v+w}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)(v + w)^i = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)v^i + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)w^i = D_v(f) + D_w(f).$$  

Since $f$ was arbitrary, this shows $D_{v+w} = D_v + D_w$, which is to say $\mathcal{C}(v + w) = \mathcal{C}(v) + \mathcal{C}(w)$. Similarly if $c$ is a constant, then

$$D_{cv}(f) = \sum_{i=1}^{n} \frac{c}{\partial x^i}(x)v^i.$$  

(b) Show that $\mathcal{C}(v) = 0$ (that is, $D_v(f) = 0$ for all $f \in C^\infty(M)$) implies that $v = 0 \in \mathbb{R}^n_x$.

**Solution:** Suppose that $D_v(f) = 0$ for all smooth functions $f$. Then this is in particular true for the coordinate functions $x^1, x^2, \ldots, x^n$. We compute for a fixed $j \in \{1, 2, \ldots, n\}$,

$$D_v(x_j) = \sum_{i=1}^{n} \frac{\partial x^i}{\partial x^j}v^i = v^j,$$

because the partial derivative $\frac{\partial x^j}{\partial x^i}$ always equals zero unless $i = j$ in which case it equals one. Thus the hypothesis that $D_v(x_j) = 0$ for each $j$ implies $v^j = 0$ for each $j$. This means precisely that $v = 0$ is the zero vector.

(c) Show that $\mathcal{C}$ is onto. That is, show that for any given $L \in T_xU$, there exists $v$ such that $D_v = L$. To do this, consider the functions $x^i : U \to \mathbb{R}$. Set $v^i = L(x^i)$, and $v = (v^1, \ldots, v^n)$. Prove that $L = D_v$.

Further hints: Taylor's theorem implies that for $y$ near $x$,

$$f(y) = f(x) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)(y^i - x^i) + \sum_{i=1}^{n} g_i(y)(y^i - x^i)$$

for some functions $g_i$ such that $g_i(x) = 0$. (In this equation, it is helpful to view $x$ as constant and $y$ as variable.) Use the fact that $L$ is linear and satisfies the Leibniz rule at $x$ together with the formula above to prove $L(f) = D_v(f)$.

**Solution:** Let us change the notation slightly from the problem statement. Let $x_0 \in U$ be the fixed point where our tangent vector lives: $L \in T_{x_0}U$. The coordinates of $x_0$ are $x_0^1, \ldots, x_0^n$, and these are constants. We will use $x^1, \ldots, x^n$ for variables/coordinates in $U$. With this notation, the Leibniz rule takes the form

$$L(fg) = L(f)g(x_0) + f(x_0)L(g)$$

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and the Taylor theorem stated in the hint takes the form

\[ f(x) = f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0) (x^i - x_0^i) + \sum_{i=1}^{n} g_i(x) (x^i - x_0^i) \]

where \( g_i(x_0) = 0 \). Note that in this equation \( f(x_0) \) and \( \frac{\partial f}{\partial x^i}(x_0) \) are constants. Now apply \( L \) to the equation. Using linearity of \( L \), we see that \( L(f) \) is the sum of several terms, namely,

\[ L(f(x_0)) + \sum_{i=1}^{n} L\left( \frac{\partial f}{\partial x^i}(x_0) \right) (x^i - x_0^i) \]

\[ + \sum_{i=1}^{n} L\left( g_i(x) \right) (x^i - x_0^i) \]

The first term \( L(f(x_0)) \) is \( L \) of a constant, and we claim this is always zero. Consider first of all the constant function 1. Then since \( 1 = 1 \cdot 1 \), the Leibniz rule implies

\[ L(1) = L(1 \cdot 1) = L(1) \cdot 1 + 1 \cdot L(1) = 2L(1) \]

Subtracting \( L(1) \) from both sides of this equation gives \( 0 = L(1) \). If \( c \) is any constant, then \( L(c) = cL(1) = c \cdot 0 = 0 \) by linearity. Thus \( L(f(x_0)) = 0 \).

For the next set of terms, we calculate

\[ L\left( \frac{\partial f}{\partial x^i}(x_0) \right) (x^i - x_0^i) = \frac{\partial f}{\partial x^i}(x_0)L\left( [x^i - x_0^i] \right) = \frac{\partial f}{\partial x^i}(x_0)[L(x^i) - L(x_0^i)] = \frac{\partial f}{\partial x^i}(x_0)L(x^i) \]

where we have used the properties of \( L \) and the fact that \( \frac{\partial f}{\partial x^i}(x_0) \) and \( x_0^i \) are constants.

For the last set of terms, we calculate

\[ L\left( g_i(x) \right) (x^i - x_0^i) = L(g_i(x))(x^i - x_0^i)|_{x=x_0} + g_i(x_0)L(x^i - x_0^i) \]

Since \( (x^i - x_0^i)|_{x=x_0} = x_0^i - x_0^i = 0 \) and \( g_i(x_0) = 0 \), this whole term is zero.

Putting it all together we find that

\[ L(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0)L(x^i). \]

The factor \( L(x^i) \) is some number that we call \( v^i \), and set \( v = (v^1, v^2, \ldots, v^n) \). Then we have

\[ L(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0)v^i = D_v(f). \]

Since this is true for any function \( f \), we see that \( L = D_v = \mathcal{E}(v) \). Since \( L \in T_{x_0} \mathcal{U} \) was arbitrary, we see that \( \mathcal{E} \) is onto.

3. Consider the map

\[ F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \]

\[ (u, v) \mapsto u \times v \]

where \( u \times v \) is the vector cross product. Compute a matrix representative of \( F \).
Solution: The formula for the cross product in components is
\[(u^1, u^2, u^3) \times (v^1, v^2, v^3) = (u^2 v^3 - u^3 v^2, u^3 v^1 - u^1 v^3, u^1 v^2 - u^2 v^1)\]

Thus the map \(F(u, v)\) has components
\[F^1(u, v) = u^2 v^3 - u^3 v^2\]
\[F^2(u, v) = u^3 v^1 - u^1 v^3\]
\[F^3(u, v) = u^1 v^2 - u^2 v^1\]

Using the standard coordinates \((u^1, u^2, u^3, v^1, v^2, v^3)\) on \(\mathbb{R}^3 \times \mathbb{R}^3\), a matrix representative of \(F^*_j\) with respect to \(u^i\) and \(v^j\). Thus
\[
[F_*] = \begin{bmatrix}
\frac{\partial F^1}{\partial u^1} & \frac{\partial F^1}{\partial u^2} & \frac{\partial F^1}{\partial u^3} & \frac{\partial F^2}{\partial u^1} & \frac{\partial F^2}{\partial u^2} & \frac{\partial F^2}{\partial u^3} & \frac{\partial F^3}{\partial u^1} & \frac{\partial F^3}{\partial u^2} & \frac{\partial F^3}{\partial u^3}
\end{bmatrix} = \begin{bmatrix}
0 & v^3 & -v^2 & 0 & -u^3 & u^2 \\
-v^3 & 0 & v^1 & u^3 & 0 & -u^1 \\
v^2 & -v^1 & 0 & -u^2 & u^1 & 0
\end{bmatrix}
\]