Math 481: Homework 2
Due Wednesday, February 12, 2020

1. (a) Propose a smooth atlas for the set $S^1 \times S^1$.
(b) Verify that one of the charts of your atlas satisfies the definition of a chart.
(c) Verify that two of your charts with intersecting domains are compatible.

2. Use the following hints to prove that for $U \subset \mathbb{R}^n$ open, the map $\mathcal{C} : \mathbb{R}^n \rightarrow T_x U$, $v \mapsto D_v$ is a bijection, where
   $$D_v(f) = \frac{d}{dt} \bigg|_{t=0} f(x + tv) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x)v^i.$$  
   (a) Show that $\mathcal{C}$ is a linear map.
   (b) Show that $\mathcal{C}(v) = 0$ (that is, $D_v(f) = 0$ for all $f \in C^\infty(M)$) implies that $v = 0 \in \mathbb{R}^n$.
   (c) Show that $\mathcal{C}$ is onto. That is, show that for any given $L \in T_x U$, there exists $v$ such that $D_v = L$. To do this, consider the functions $x^i : U \rightarrow \mathbb{R}$. Set $v^i = L(x^i)$, and $v = (v^1, \ldots, v^n)$. Prove that $L = D_v$.

   Further hints: Taylor’s theorem implies that for $y$ near $x$,
   $$f(y) = f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x)(y^i - x^i) + \sum_{i=1}^n g_i(y)(y^i - x^i)$$
   for some functions $g_i$ such that $g_i(x) = 0$. (In this equation, it is helpful to view $x$ as constant and $y$ as variable.) Use the fact that $L$ is linear and satisfies the Leibniz rule at $x$ together with the formula above to prove $L(f) = D_v(f)$.

3. Consider the map
   $$F : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$
   $$(u, v) \mapsto u \times v$$
   where $u \times v$ is the vector cross product. Compute a matrix representative of $F_*$.