1. (a) Use the Regular Value Theorem to prove that the set
\[ \{(x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2 + y^2 - 2}\right)^2 + z^2 = 1\} \]
is an embedded submanifold of \( \mathbb{R}^3 \).

**Solution:** This set is \( F^{-1}(1) \), where \( F : \mathbb{R}^3 \to \mathbb{R} \) is the function
\[ F(x, y, z) = \left(\sqrt{x^2 + y^2 - 2}\right)^2 + z^2 \]
The matrix of the derivative \([F_*]\) is nothing but the row vector of partial derivatives of \( F \):
\[ [F_*] = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} \]
\[ = \begin{bmatrix} 2(\sqrt{x^2 + y^2 - 2})(1/2)(x^2 + y^2)^{-1/2}x & 2(\sqrt{x^2 + y^2 - 2})(1/2)(x^2 + y^2)^{-1/2}y & 2z \end{bmatrix} \]
\[ = \begin{bmatrix} 2(\sqrt{x^2 + y^2 - 2})(x^2 + y^2)^{-1/2}x & 2(\sqrt{x^2 + y^2 - 2})(x^2 + y^2)^{-1/2}y & 2z \end{bmatrix} \]
In these formulas, we notice that the first and second components are undefined if \( x = y = 0 \), which is to say along the \( z \)-axis. This means that the function \( F \) is not smooth along the \( z \)-axis. Fortunately, the set \( F^{-1}(1) \) that we are interested in does not intersect the \( z \)-axis, for if \( x = y = 0 \), then \( F(0, 0, z) = (-2)^2 + z^2 = z^2 + 4 \geq 1 \), so \( (0, 0, z) \) is not in \( F^{-1}(1) \).
Next, let’s look for critical points outside of the \( z \)-axis and see if they lie on \( F^{-1}(1) \). Since we are assuming that \( x \) and \( y \) are not both zero, the only way for the first and second components of \([F_*]\) to be zero is if \( \sqrt{x^2 + y^2 - 2} = 0 \). The third component is zero if and only if \( z = 0 \). So the critical points are where \( \sqrt{x^2 + y^2} = 2 \) and \( z = 0 \). At such points, we see \( F(x, y, z) = 0^2 + 0^2 = 0 \neq 1 \), so no critical points are in the set \( F^{-1}(1) \). Thus 1 is a regular value of \( F \), and the Regular Value Theorem implies that \( F^{-1}(1) \) is an embedded submanifold of \( \mathbb{R}^3 \).

[For those of you who want to be really pedantic, you might argue that the regular value theorem does not apply to \( F : \mathbb{R}^3 \to \mathbb{R} \) because this is not a smooth map. To get around this, we define \( M = \mathbb{R}^3 \setminus \{(0,0,z) \mid z \in \mathbb{R}\} \) to be the complement of the \( z \)-axis. Then \( M \) is an open subset of \( \mathbb{R}^3 \) and hence a manifold. The function \( F \) with domain restricted to \( M \) does define a smooth map \( \tilde{F} : M \to \mathbb{R} \). The arguments above show that the set we are interested in is a subset of \( M \) and so \( \tilde{F}^{-1}(1) = F^{-1}(1) \). The the regular value theorem implies that \( \tilde{F}^{-1}(1) \) is an embedded submanifold of \( M \). Since \( M \subset \mathbb{R}^3 \) is open it follows that \( \tilde{F}^{-1}(1) \) is a submanifold of \( \mathbb{R}^3 \).]

(b) Draw the set. *Hint:* Use cylindrical coordinates \((r, \theta, z)\) on \( \mathbb{R}^3 \).

**Solution:** In cylindrical coordinates, the equation for the set becomes \((r - 2)^2 + z^2 = 1\). Since this equation does not involve the angular variable \( \theta \), the set is rotationally symmetric about the \( z \)-axis. In fact it is the surface of revolution obtained by taking the circle \((x - 2)^2 + z^2 = 1\) in the \( xz \)-plane (with center \((2, 0)\) and radius 1) and revolving around the \( z \)-axis.
2. Consider the function

\[ f : \mathbb{R}^2 \to \mathbb{R} \]
\[ [x^1, x^2, x^3] \mapsto \frac{3(x^1)^2}{(x^1)^2 + (x^2)^2 + (x^3)^2} \]

(a) Find all of the critical points of \( f \). Hint: You need to check in all three of our standard coordinate charts \( ((U_i, \phi_i))_{i=1,2,3} \).

**Solution:** The first coordinate chart \((U_1, \phi_1)\) is \( U_1 = \{[x^1, x^2, x^3] \mid x^1 \neq 0\}, \phi_1([x^1, x^2, x^3]) = (x^2/x^1, x^3/x^1), \) with inverse \( \phi_1^{-1}(u, v) = [1, u, v]. \) The coordinate representation of \( f \) in this chart is

\[ f \circ \phi_1^{-1}(u, v) = f([1, u, v]) = \frac{3}{1 + u^2 + v^2}. \]

For each \( p \in U_1 \), the chart gives rise to a basis \( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \) of \( T_p \mathbb{R}^2 \). With respect to this basis, the matrix for the derivative \([f_*]\) is given by the row vector of partial derivatives of \( f \circ \phi_1^{-1} \) with respect to \( u \) and \( v \).

\[ [f_*] = \begin{bmatrix} \frac{\partial}{\partial u} (f \circ \phi_1^{-1}) & \frac{\partial}{\partial v} (f \circ \phi_1^{-1}) \end{bmatrix} = \begin{bmatrix} -3(1 + u^2 + v^2) & 2u \\ -3(1 + u^2 + v^2) & 2v \end{bmatrix} \]

The critical points are where this matrix is zero; since the quantity \( 1 + u^2 + v^2 \) is never zero, this only happens when \( u = v = 0 \). This corresponds to the point \( \phi_1^{-1}(0, 0) = [1, 0, 0] \in U_1 \). Thus \([1, 0, 0]\) is the only critical point in \( U_1 \).

The second coordinate chart is \( U_2 = \{[x^1, x^2, x^3] \mid x^2 \neq 0\}, \phi_2([x^1, x^2, x^3]) = (x^1/x^2, x^3/x^2), \) with inverse \( \phi_2^{-1}(u, v) = [u, 1, v] \) (here we are reusing the symbols \( u, v \) with different meaning). The coordinate representation of \( f \) in this chart is

\[ f \circ \phi_2^{-1}(u, v) = f([u, 1, v]) = \frac{3u^2}{u^2 + 1 + v^2}. \]

Again, the matrix \([f_*]\) contains the partial derivatives of this expression, so

\[ [f_*] = \begin{bmatrix} \frac{1 + u^2 + v^2}{(1 + u^2 + v^2)^2} & -3u^2 \\ \frac{1 + u^2 + v^2}{(1 + u^2 + v^2)^2} & 2v \end{bmatrix} = \begin{bmatrix} \frac{1 + u^2 + v^2}{(1 + u^2 + v^2)^2} & -3u^2 \\ \frac{1 + u^2 + v^2}{(1 + u^2 + v^2)^2} & 2v \end{bmatrix} \]

The condition that both derivatives be zero amounts to \( (1 + v^2)u = 0 \) and \( u^2v = 0 \). If \( u \neq 0 \), this forces \( 1 + v^2 = 0 \) and \( v = 0 \), which are contradictory conditions. So we must have \( u = 0 \), in which case \( v \) can be anything. Thus the critical points have coordinates \((0, v)\) for any \( v \in \mathbb{R} \), and the corresponding points are \([0, 1, v] \in U_2 \) for all \( v \in \mathbb{R} \).

The third coordinate chart is \( U_3 = \{[x^1, x^2, x^3] \mid x^3 \neq 0\}, \phi_3([x^1, x^2, x^3]) = (x^1/x^3, x^2/x^3), \) with inverse \( \phi_3^{-1}(u, v) = [u, v, 1] \). The coordinate representation of \( f \) is

\[ f \circ \phi_3^{-1}(u, v) = f([u, v, 1]) = \frac{3u^2}{u^2 + v^2 + 1}. \]

This is formally the same as the previous case, so we know the critical points are where \( u = 0 \) and \( v \in \mathbb{R} \). The corresponding points in \( U_3 \) are \([0, v, 1] \) for \( v \in \mathbb{R} \).

Thus, the critical points are \([1, 0, 0]\), and also \([0, 1, v]\) and \([0, v, 1]\) for any \( v \in \mathbb{R} \).

An optional remark is that the second and third types of critical points can be described as a single set, namely as the set of all points of the form \([0, x^2, x^3]\) where \( x^2 \) and \( x^3 \) are
not both zero. Since all three coordinates cannot be zero for a point in \( \mathbb{RP}^2 \), this is the same as the set of points that have first coordinate zero. For any such point, we can either normalize the second or third coordinate to be 1.

Another optional remark is that, thinking of \( \mathbb{RP}^2 \) as the set of lines through the origin in \( x^1x^2x^3\)-space, the set of critical points of \( f \) consists of the \( x^1\)-axis, and also all lines contained in the \( x^2x^3\)-plane.

(b) Find the point at which \( f \) takes its maximum value.

Solution: Although this was not mentioned in class, it is true that, for a smooth function on a manifold such as \( \mathbb{RP}^2 \) (a compact manifold), the maximum and minimum always exist and occur at critical points. Since we computed the critical points of \( f \) in the previous part, we can simply plug those points into \( f \) and see which value is largest.

\[
f([1,0,0]) = \frac{3}{1 + 0 + 0} = 3,
\]
\[
f([0,x^2,x^3]) = \frac{0}{0 + (x^2)^2 + (x^3)^2} = 0.
\]

Thus the critical values of \( f \) are 0 and 3. Since the maximum must be a critical value, the maximum is 3. Incidentally, the minimum value is 0.

(c) Find all values \( c \) for which the Regular Value Theorem implies that \( f^{-1}(c) \) is a manifold.

Solution: The Regular Value Theorem asserts that \( f^{-1}(c) \) is a manifold as long as \( c \) is a regular value, that is, not a critical value. Since we computed the critical values in the previous part, we see that the set of regular values is \( \mathbb{R} \setminus \{0,3\} \). For any \( c \in \mathbb{R} \setminus \{0,3\} \), we know that \( f^{-1}(c) \) is a manifold.

An optional remark is that if \( c < 0 \) or \( c > 3 \), then \( f^{-1}(c) \) is the empty set. (We regard the empty set as a manifold of any dimension.)