Math 481: Homework 4 Solutions
Due Friday, February 28, 2020

1. Solve the Initial Value Problems encoded by the following vector fields on \( \mathbb{R}^2 \). Write your solution in the form of a map \( \phi_t : \mathbb{R}^2 \to \mathbb{R}^2 \).

(a) \( V(x) = \begin{vmatrix} x^2 \left( \frac{\partial}{\partial x^1} \right)_x \frac{\partial}{\partial x} \\ -x^1 \left( \frac{\partial}{\partial x^2} \right)_x \frac{\partial}{\partial x^1} \end{vmatrix} \). 

Solution: Let us denote a solution curve by \( \gamma(t) = (\gamma^1(t), \gamma^2(t)) \). The differential equations are

\[
\begin{align*}
\frac{d}{dt}(\gamma^1(t)) &= V^1(\gamma(t)) = \gamma^2(t) \\
\frac{d}{dt}(\gamma^2(t)) &= V^2(\gamma(t)) = -\gamma^1(t)
\end{align*}
\]

To simplify notation let’s write \( u(t) = \gamma^1(t) \) and \( v(t) = \gamma^2(t) \). Then the equations are

\[
\begin{align*}
u' &= v \\
v' &= -u
\end{align*}
\]

where prime denotes derivative with respect to \( t \). To solve this system, observe that if we can find \( u \) then \( v \) is determined by \( v = u' \). To get an equation for \( u \) alone, differentiate the first equation to get \( u'' = v' \), and combine with the second to get

\[
u'' = -u
\]

This is a second-order constant-coefficient linear homogeneous ordinary differential equation, whose general solution is

\[
u = A \cos(t) + B \sin(t)
\]

where \( A \) and \( B \) are undetermined constants. [See Chapter 2 of Elementary Differential Equations by Edwards and Penney, or any other book on ODE.] Thus

\[
v = u' = -A \sin(t) + B \cos(t)
\]

To determine the constants, we use an initial condition. Suppose this initial condition is \( x_0 = (x_0^1, x_0^2) \). The we set

\[
(x_0^1, x_0^2) = \gamma(0) = (u(0), v(0)) = (A, B)
\]

Let us call \( \gamma_{x_0}(t) \) this particular solution. Then we have

\[
\gamma_{x_0}(t) = (x_0^1 \cos(t) + x_0^2 \sin(t), -x_0^1 \sin(t) + x_0^2 \cos(t)).
\]

The map \( \phi_t : \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by \( \phi_t(x) = \gamma_x(t) \), so

\[
\phi_t(x) = (x^1 \cos(t) + x^2 \sin(t), -x^1 \sin(t) + x^2 \cos(t))
\]
(b) \( W(x) = (x^1)^2 \frac{\partial}{\partial x^1} x + 3x^2 \frac{\partial}{\partial x^2} x \). (Is the solution of \( W \) defined for all \( t \in \mathbb{R} \)?)

**Solution:** As before, let us suppose \( \gamma(t) = (\gamma^1(t), \gamma^2(t)) \) is a solution of the system

\[
\frac{d}{dt}(\gamma^1(t)) = V^1(\gamma(t)) = [\gamma^1(t)]^2 \\
\frac{d}{dt}(\gamma^2(t)) = V^2(\gamma(t)) = 3\gamma^2(t)
\]

With \( u = \gamma^1 \) and \( v = \gamma^2 \), this becomes the system

\[
u' = u^2 \\
\nu' = 3v
\]

Observe that these equations are not coupled, so we can solve for \( u \) and \( v \) separately. The second equation is linear homogeneous, and the general solution is

\[v = Be^{3t}\]

where \( B \) is an undetermined constant. For the first equation, we use separation of variables. \( du/dt = u^2 \) becomes

\[
\frac{du}{u^2} = dt \\
\int \frac{du}{u^2} = \int dt \\
-u^{-1} = t + C \\
u = (-t - C)^{-1}
\]

The initial condition \( \gamma(0) = (x_0^1, x_0^2) \) allows us to determine \( B \) and \( C \) as

\[
x_0^1 = u(0) = -C^{-1} \\
C = -1/x_0^1 \\
x_0^2 = v(0) = B
\]

Thus the particular solution is

\[\gamma_{x_0}(t) = ((-t + 1/x_0^1)^{-1}, x_0^2 e^{3t}) = \left( \frac{x_0^1}{1-x_0^1 t}, x_0^2 e^{3t} \right)\]

The map \( \phi_t \) is given by

\[\phi_t(x) = \gamma_x(t) = \left( \frac{x^1}{1-x^1 t}, x^2 e^{3t} \right)\]

Observe that the map \( \phi_t(x) \) is not defined if \( 1-x^1 t = 0 \), which is to say, \( t = 1/x^1 \). By plotting the first component function of \( \phi_t \) for several values of \( x^1 \), we can see that what happens is the following: if the initial condition \( x^1 \) is positive, then the solution curve exists for all negative times as well as all positive times up to \( t = 1/x^1 \), at which point the solution becomes infinite. If the initial condition \( x^1 \) is negative, then the solution curve exists for all positive times, but if we extrapolate the solution backwards in time it becomes infinite at \( t = 1/x^1 < 0 \). If the initial condition is \( x^1 = 0 \), the the solution exists for all time, in fact it stays put at \( x^1 = 0 \).
2. Prove that for any smooth manifold \( M \) the map \( \pi : TM \to M \) is smooth.

**Solution:** The trick for this problem is that we can check smoothness of a map using any charts we wish, so we will use charts that make the map \( \pi \) as simple as possible. First we recall how the charts on \( TM \) are determined by the charts on \( M \). Let \((U_a, \phi_a)\) be a chart from the atlas for \( M \). Then, for any \( p \in U_a \), the chart determines a basis for the tangent space \( T_pM \), namely

\[
\frac{\partial}{\partial x^1}_p, \ldots, \frac{\partial}{\partial x^n}_p
\]

where \((x^1, \ldots, x^n)\) denote the \((U_a, \phi_a)\)-coordinates. Then any vector \( v \in T_pM \) can be written uniquely as

\[
v = v^1 \frac{\partial}{\partial x^1}_p + \cdots + v^n \frac{\partial}{\partial x^n}_p
\]

for some \( n \)-tupel of numbers \((v^1, \ldots, v^n)\). Let us denote this \( n \)-tupel as \([v]_a, p\); it is the components of \( v \) when written in the basis of \( T_pM \) coming from the coordinate chart \((U_a, \phi_a)\). With this notation, we have a chart \((\hat{U}_a, \hat{\phi}_a)\) on \( TM \) given by

\[
\hat{U}_a = \pi^{-1}(U_a) = \{(p, v) \mid p \in U_a, v \in T_pM\},
\]

\[
\hat{\phi}_a(p, v) = (\phi_a(p), [v]_a, p) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}
\]

Observe that the inverse of \( \hat{\phi}_a \) is

\[
\hat{\phi}_a^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = \left( \phi_a^{-1}(x^1, \ldots, x^n), \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}_p \right)
\]

where \((x^1, \ldots, x^n) \in \phi_a(U_a)\).

Finally, let’s consider the map \( \pi : TM \to M, \pi(p, v) = v \). We want to check smoothness of this map at a given point \((p_0, v_0)\) in the domain. So choose an index \( a \) such that \((\hat{U}_a, \hat{\phi}_a)\) contains \((p_0, v_0)\). Then \((U_a, \phi_a)\) is a chart on the target (that is, \( M \)) that contains \( \pi(p_0, v_0) = p_0 \). Thus we may check smoothness of \( \pi \) at \((p_0, v_0)\) using the chart \((\hat{U}_a, \hat{\phi}_a)\) on the domain and the chart \((U_a, \phi_a)\) on the target. Thus we must check smoothness of the map

\[
\phi_a \circ \pi \circ \hat{\phi}_a : \hat{\phi}_a(\hat{U}_a) \to \phi_a(U_a)
\]

which is a map from an open subset of \( \mathbb{R}^{2n} \) to a subset of \( \mathbb{R}^n \). The formula for this composition is actually very simple

\[
\phi_a \circ \pi \circ \hat{\phi}_a(x^1, \ldots, x^n, v^1, \ldots, v^n)
= \phi_a \left( \phi_a^{-1}(x^1, \ldots, x^n), \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}_p \right)
= \phi_a(\phi_a^{-1}(x^1, \ldots, x^n))
= (x^1, \ldots, x^n)
\]

In short, the map is \((x^1, \ldots, x^n, v^1, \ldots, v^n) \to (x^1, \ldots, x^n)\), that is, the map that deletes the last \( n \) coordinates. This is a linear map in these coordinates, hence it is smooth.
3. Let $M$ be a two-dimensional manifold. Suppose there are two vector fields on $M$, say $V$ and $W$, such that $\text{Span}\{V(p), W(p)\} = T_p M$ for all $p \in M$. Construct a diffeomorphism from $TM$ to $M \times \mathbb{R}^2$.

**Solution:** Because $M$ is two-dimensional, the tangent space $T_p M$ is a two-dimensional vector space for any $p \in M$. Because $V(p)$ and $W(p)$ are a pair of vectors that span this space, they are moreover a basis of $T_p M$. This means that any vector $v \in T_p M$ can be written in exactly one way as

$$v = sV(p) + tW(p)$$

for a pair of numbers $(s, t)$. We introduce the notation $[v]_{(V(p), W(p))}$ for this pair of components of $v \in T_p M$ with respect to the basis $\{V(p), W(p)\}$. Then we define a map $F: TM \to M \times \mathbb{R}^2$ by

$$F(p, v) = (p, [v]_{(V(p), W(p))})$$

We also define a map $G: M \times \mathbb{R}^2 \to TM$ by

$$G(p, a, b) = (p, aV(p) + bW(p))$$

We claim that $F$ and $G$ are inverse functions of each other. The composition $F \circ G$ is

$$F(G(p, a, b)) = (p, [aV(p) + bW(p)]_{(V(p), W(p))})$$

The components of $aV(p) + bW(p)$ with respect to the basis $\{V(p), W(p)\}$ are evidently $a$ and $b$, so

$$F(G(p, a, b)) = (p, a, b)$$

Thus $F \circ G$ is the identity. For the composition $G \circ F$, we need to show that

$$G(F(p, v)) = G(p, [v]_{(V(p), W(p))}) = (p, w)$$

is equal to $(p, v)$; here $w \in T_p$ the linear combination $w = sV(p) + tW(p)$ where $(s, t) = [v]_{(V(p), W(p))}$ are the components of $v$ with respect to the basis $\{V(p), W(p)\}$. But it is evident that $w$ and $v$ have the same components with respect to the basis $\{V(p), W(p)\}$, so $w = v$. This completes the proof that $F$ and $G$ are inverses of each other.

It remains to show that $F$ and $G$ are smooth maps. For this, we must choose charts on $TM$ and $M \times \mathbb{R}^2$. Each chart $(U, \phi)$ on $M$ gives rise to a chart $(\hat{U}, \hat{\phi})$ on $TM$ as described in the previous problem. It also gives rise to a chart $(U \times \mathbb{R}^2, \psi)$ on $M \times \mathbb{R}^2$, where $\psi(p, a, b) = (\phi(p), a, b)$ for all $p \in U, a \in \mathbb{R}, b \in \mathbb{R}$. To show smoothness of $F$, we consider the composition

$$\psi \circ F \circ \hat{\phi}^{-1}(x^1, x^2, v^1, v^2)$$

$$= \psi \circ F \left(\phi^{-1}(x^1, x^2), v^1 \frac{\partial}{\partial x^1} \bigg|_p + v^2 \frac{\partial}{\partial x^2} \bigg|_p \right)$$

$$= \psi \left(\phi^{-1}(x^1, x^2), \left[ v^1 \frac{\partial}{\partial x^1} \bigg|_p + v^2 \frac{\partial}{\partial x^2} \bigg|_p \right]_{(V(p), W(p))} \right)$$

$$= \left(\phi(\phi^{-1}(x^1, x^2)), \left[ v^1 \frac{\partial}{\partial x^1} \bigg|_p + v^2 \frac{\partial}{\partial x^2} \bigg|_p \right]_{(V(p), W(p))} \right)$$
\[
\left( x^1, x^2, \left[ \frac{\partial}{\partial x^1} p + \frac{\partial}{\partial x^2} p \right] \right)_{(V(p), W(p))}
\]

In this and subsequent calculations, we are abbreviating \( p = \phi^{-1}(x^1, x^2) \). To show that this expression is a smooth function, the central point is to understand why

\[
\left[ \frac{\partial}{\partial x^1} p + \frac{\partial}{\partial x^2} p \right] \bigg|_{(V(p), W(p))}
\]

is a smooth function of \((x^1, x^2, v^1, v^2)\).

For this we need to understand the relationships between the bases \(\{V(p), W(p)\}\) and \(\left\{ \frac{\partial}{\partial x^1} p, \frac{\partial}{\partial x^2} p \right\}\), which are both bases of \(T_p M\). Because \(V\) and \(W\) are vector fields, we can write

\[
V(p) = V^1(p) \frac{\partial}{\partial x^1} p + V^2(p) \frac{\partial}{\partial x^2} p,
\]

\[
W(p) = W^1(p) \frac{\partial}{\partial x^1} p + W^2(p) \frac{\partial}{\partial x^2} p,
\]

where \(V^1, V^2, W^1, W^2\) are functions defined in \(U\). Because \(V\) and \(W\) are smooth vector fields, these functions are smooth. Now suppose \(v \in T_p M\) is some vector that is represented in the basis \(\{V(p), W(p)\}\) as

\[
v = aV(p) + bW(p),
\]

and represented in the \(\left\{ \frac{\partial}{\partial x^1} p, \frac{\partial}{\partial x^2} p \right\}\) basis as

\[
v = v^1 \frac{\partial}{\partial x^1} p + v^2 \frac{\partial}{\partial x^2} p.
\]

Equating these two expressions and using the relations between the two bases, we find

\[
[aV^1(p) + bW^1(p)] \frac{\partial}{\partial x^1} p + [aV^2(p) + bW^2(p)] \frac{\partial}{\partial x^2} p = v^1 \frac{\partial}{\partial x^1} p + v^2 \frac{\partial}{\partial x^2} p,
\]

implying that corresponding coefficients are equal:

\[
v^1 = aV^1(p) + bW^1(p), \quad v^2 = aV^2(p) + bW^2(p)
\]

In matrix form this system is

\[
\begin{bmatrix}
v^1 \\
v^2
\end{bmatrix} =
\begin{bmatrix}
V^1(p) & W^1(p) \\
V^2(p) & W^2(p)
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
\]

Because \(V(p)\) and \(W(p)\) are a basis, the \(2 \times 2\) matrix in this equation is invertible, and the inverse relation is

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} = \frac{1}{V^1(p)W^2(p) - W^1(p)V^2(p)}
\begin{bmatrix}
W^2(p) & -W^1(p) \\
-V^2(p) & V^1(p)
\end{bmatrix}
\begin{bmatrix}
v^1 \\
v^2
\end{bmatrix}
\]

Now we realize that the \(a, b\) here are precise what we are looking for when we compute

\[
\left[ \frac{\partial}{\partial x^1} p + \frac{\partial}{\partial x^2} p \right] \bigg|_{(V(p), W(p))}
\]
To spell it out, we have
\[
\left[ v^1 \frac{\partial}{\partial x^1} + v^2 \frac{\partial}{\partial x^2} \right]_{(p)} = \left( \frac{v^1 W^2(p) - v^2 W^1(p)}{V^1(p) W^2(p) - W^1(p) V^2(p)}, \frac{-v^1 V^2(p) + v^2 V^1(p)}{V^1(p) W^2(p) - W^1(p) V^2(p)} \right)
\]

We claim that these quantities depend smoothly on \((x^1, x^2, v^1, v^2)\): they are smooth with respect to \(v^1\) and \(v^2\) because they are linear in these variables, and they are smooth with respect to \((x^1, x^2)\) because \(V^1, V^2, W^1, W^2\) depend smoothly on \((x^1, x^2)\), and the expression is a rational function of these quantities whose denominator is never zero. This completes the proof that \(F\) is smooth.

The proof that \(G\) is smooth is similar, and a bit easier. We must check smoothness of
\[
\hat{\phi} \circ G \circ \psi^{-1}(x^1, x^2, a, b)
\]
\[
= \hat{\phi} \circ G(\phi^{-1}(x^1, x^2), a, b)
\]
\[
= \hat{\phi}(\phi^{-1}(x^1, x^2, aV(p) + bW(p))
\]
\[
= \left( x^1, x^2, [aV(p) + bW(p)] \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right)
\]
\[
= (x^1, x^2, aV^1(p) + bW^1(p), aV^2(p) + bW^2(p))
\]

This is smooth because all components are polynomial combinations of the coordinates and the smooth functions \(V^1, V^2, W^1, W^2\).