let $M$ be a set.

eg $M = \{ \text{possible configurations of a robot} \}$

Q. When can define what it means for $f: M \to \mathbb{R}$ to be continuous or smooth?

Def. A coordinate chart on $M$ is a subset $U \subset M$ and a map $\phi_u: U \to \mathbb{R}^n$ such that $\phi_u(U) \subset \mathbb{R}^n$ is open and $\phi_u$ is 1-1.

Remark. Let $\Pi_i: \mathbb{R}^n \to \mathbb{R}$ be projection $\Pi_i(x_1, \ldots, x^n) = x_i$.

Given $(U, \phi_u)$ we get a local coordinate function

$x_u^i: U \to \mathbb{R}$

$p \mapsto \Pi_i \circ \phi_u(p)$
\[ M = \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \} \]
\[ = \left\{ (x_1, x_2, x_3) \mid \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = 1 \right\} \]
= unit sphere, \( S^2 \)

Let \( U = \{ x \in S^2 \mid x^3 > 0 \} \) (Northern hemisphere)

and \( \phi_u : U \longrightarrow \mathbb{R}^2 \quad )
\[ (x_1, x_2, x_3) \rightarrow (x_1, x_2) \]

Claim \( \phi_u \) is a coordinate chart

\[ \phi_u(U) = \{ (x_1, x_2) \mid (x_1)^2 + (x_2)^2 = 1 \} \]
\[ = \{ (x_1, x_2) \mid (x_1)^2 + (x_2)^2 < 1 \} \]
\[ = B^2_1((0,0)) \quad \text{which is open} \]

Need \( \phi_u \) to be 1-1.

Suppose \( x, y \in U \) and \( \phi_u(x) = \phi_u(y) \).
\[ x = (x', x^2, \sqrt{1 - (x')^2 - (x^2)^2}) \quad \text{and} \quad (x', x^2) = (y', y^2). \]

\[ x = y. \]

Note \( \phi_u^{-1}(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \)

\textbf{Attempt 1} \quad f : M \to \mathbb{R} \text{ is smooth near } p \in M \text{ if there is a chart } (U, \phi_u) \text{ with } p \in U \text{ such that } f \circ \phi_u^{-1} : \phi_u(U) \to \mathbb{R} \text{ is smooth.}

\textbf{Problems}

P1) Need a chart around (each) \( p \in M. \)

P2) Smoothness near \( p \) should be independent of choice of this chart.

\textbf{Def.} Two charts \((U, \phi_u)\) and \((V, \phi_v)\) on \( M \) are (smoothly) compatible if either \( U \cap V = \emptyset \) or if \( \phi_u(U \cap V) \subset \mathbb{R}^n \) is open and \( \phi_v \circ \phi_u^{-1} : \phi_u(U \cap V) \to \mathbb{R}^n \) is smooth.
A smooth atlas on $M$ is a collection of charts $\mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}_{\alpha \in A}$ such that

I) $\bigcup_{\alpha \in A} U_\alpha = M$

II) Any two charts in $\mathcal{A}$ are compatible

"Def" a smooth manifold (of dimension $n$) is a set $M$ equipped with a smooth atlas $\mathcal{A}$ (where each
A function $f: M \to \mathbb{R}$ is **smooth** near $p$ (w.r.t. $\mathcal{A}$) if for any $(U_p, \phi_p) \in \mathcal{A}$ with $p \in U_p$ the function

$$f \circ \phi_p^{-1} : \phi_p(U_p) \to \mathbb{R}$$

is smooth.

**Proof**. Suppose $p \in U_\alpha \cap U_\beta$ and $f \circ \phi_\alpha^{-1}$ is smooth. We need $f \circ \phi_\beta^{-1}$ to also be smooth.

$$f \circ \phi_\beta^{-1} = f \circ (\phi_\alpha^{-1} \circ \phi_\beta) \circ \phi_\beta^{-1}$$
\[ = (f \circ \phi^{-1}_x) \circ (\phi_x \circ \phi^{-1}_e) \]

\[
\uparrow \quad \uparrow \\
\text{smooth by} \quad \text{smooth by} \\
\text{assumption} \quad \Pi
\]

Exercise  The composition of smooth maps is smooth.

This follows from chain rule and properties of continuity.