Def\(^3\) A smooth atlas on the set \(M\) is a collection of charts \(\mathcal{A} = \bigcup_{\alpha \in \mathcal{A}} (U_\alpha, \phi_\alpha)\) such that

1) \(\bigcup_{\alpha \in \mathcal{A}} U_\alpha = M\)

II) Any two charts in \(\mathcal{A}\) are compatible.

\( (\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^n \text{ smooth} ) \)

( Assumed that \(\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\) for fixed \(n\). )

Def\(^3\) (provisional)

A smooth manifold is a set \(M\) equipped with a smooth atlas \(\mathcal{A}\).

\(\text{Rank} \quad \dim (M, \mathcal{A}) = n\)

\(\text{Rank} \quad \text{Not all sets admit smooth atlases!}\)
Def. Given \((M, \alpha)\), a function \(f: M \to \mathbb{R}\) is smooth near \(p\) if for some \((U, \phi) \in \alpha\) with \(p \in U\) the map \(f \circ \phi^{-1}: \phi(U) \to \mathbb{R}\) is smooth near \(\phi(p)\).

Examples of “smooth manifolds”.

Ex. 0. \(U \subset \mathbb{R}^n\) open

Let \(\text{Id}_U : U \to \mathbb{R}^n\)

\[ x \mapsto x \]

Then \(\alpha = \{ (U, \text{Id}_U) \}\) is a smooth atlas.

Ex. 1. For \(f: U \subset \mathbb{R}^n \to \mathbb{R}\) continuous, define

\[ \Gamma_f = \{ (x, f(x)) \mid x \in U \}\subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}. \]
Let \( \Pi: \Gamma_f \rightarrow \mathbb{R}^n \)
\( (x, f(x)) \rightarrow x \)

Claim \( \mathcal{A} = \{ (\Gamma_f, \Pi) \} \) is a smooth atlas.

\[ \text{pf} \]
I) \( \Gamma_f = \Gamma_f \quad (\bigcup U_j = M) \)

II) \( \Pi \circ \Pi^{-1} : \Pi(\Gamma_f) \rightarrow \mathbb{R}^n \)

is the map \( \text{Id}_U \) which is smooth.

Ex.2 \( M = \{ x \in \mathbb{R}^3 \mid \| x \| = 1 \} = S^2 \)

Recall our previous chart
\( U = \{ x \in S^2 \mid x^3 > 0 \} \quad \phi_U(x) = (x', x^2) \)
rename this \( (U_3^+, \phi_3^+) \).

Define \( U_3^- = \{ x \in S^2 \mid x_3 < 0 \} \)

and \( \phi_3^- : U_3^- \rightarrow \mathbb{R}^2 \)
\( (x', x^2, x_3) \rightarrow (x', x^2) \)
Define \((U_i^+, \phi_i^+)\) and \((U_i^-, \phi_i^-)\) analogously.

Claim: \[ \mathcal{A} = \left\{ (U_i^+, \phi_i^+) \right\}_{i=1,2,3} \] is a smooth atlas on \(S^2\).

If

Each \((U_i^+, \phi_i^+)\) is a chart.

- Need \(U_i^+\) to cover \(S^2\).

\[ x \in S^2 \iff \| x \| = 1 \]
\[ \Rightarrow x_i \neq 0 \text{ for some } i \]
\[ \Rightarrow x \in U_i^+ \text{ or } U_i^- \]

- Need compatibility

For \(\phi_i^+\) and \(\phi_i^-\) we have \(U_i^+ \cap U_i^- = \emptyset\)

Need to check compatibility of \(\phi_i^+\) and \(\phi_j^+\) for \(i \neq j\).

Consider \(\phi_i^+\) and \(\phi_2^-\).

Need \(\phi_2^- (U_i^+ \cap U_i^-)\) to be open.
\[ \phi_2 \left( U_1^+ \cap U_2^- \right) = \phi_2 \left( \{ x \in \mathbb{R}^2 \mid x_1 > 0, x_2 < 0 \} \right) \]
\[ = \{ (x_1, x_2) \in B_1^2(0,0) \mid x_1 > 0, x_2 < 0 \} \]
\[ = \{ (u,v) \in B_1^2(0,0) \mid u > 0 \} \]

Exercise: this is an open subset of \( \mathbb{R}^2 \).

Need \( \phi_1^+ \circ (\phi_2^-)^{-1} \) to be smooth on \( \phi_2^- \left( U_1^+ \cap U_2^- \right) \).

\[ \phi_1^+ \circ (\phi_2^-)^{-1} (u,v) = \phi_1^+ \left( u, -\sqrt{1-u^2-v^2}, v \right) \]
\[ = \left( -\sqrt{1-u^2-v^2}, v \right) \]

This is smooth on the domain of \( \phi_1^+ \circ (\phi_2^-)^{-1} \),
where \( u^2 + v^2 < 1 \).

Exercise. For any \( n \geq 1 \) set
\[ S^n = \{ x \in \mathbb{R}^{n+1} \mid \| x \| = 1 \} \]
Define a smooth atlas \( \mathcal{A} = \{ (U_i^+, \phi_i^+) \}_{i=1, \ldots, k+1} \).

**Example (Products)**

Let \( \mathcal{A} = \{ (U_u, \phi_u) \}_{u \in A} \) be a smooth atlas on \( M \) and \( \mathcal{B} = \{ (V_v, \psi_v) \}_{v \in B} \) be a smooth atlas on \( N \).

On \( M \times N = \{ (p, q) \mid p \in M, q \in N \} \) we can define

\[
\mathcal{A} \times \mathcal{B} = \left\{ (U_u \times V_v, \phi_u \times \psi_v) \right\}
\]

where \( \phi_u \times \psi_v : (p, q) = (\phi_u(p), \psi_v(q)) \in \mathbb{R}^m \times \mathbb{R}^n \).

**Exercise** \( \mathcal{A} \times \mathcal{B} \) is a smooth atlas on \( M \times N \).

E.g. \( S^1 \times S^1 \), \( S^2 \times S^1 \), \( S^1 \times S^2 \), \( (S^1 \times S^1) \times S^1 \), …

...can all be equipped with smooth atlases.
EXAMPLE

\[ \mathbb{RP}^2 = \left\{ \text{all lines through origin in } \mathbb{R}^3 \right\} \]

\[ = \left\{ x \in \mathbb{R}^3 \setminus \{ \text{origin} \} \right\} / \sim \]

where \( x \sim y \iff x = \lambda y \text{ for } \lambda \neq 0. \)

\[ = \left\{ [x', x^2, x^3] \right\} \text{ homogeneous coords.} \]

Here \( [x', x^2, x^3] \) stands for the equivalence class of \( x = (x', x^2, x^3). \) So \( [x', x^2, x^3] = [\lambda x', \lambda x^2, \lambda x^3] \) for any \( \lambda \neq 0. \)

Note \( \mathbb{RP}^2 \) is comprised of subsets of \( \mathbb{R}^3 \) but does not live in \( \mathbb{R}^3 \) itself.

Let's construct a smooth atlas on \( \mathbb{RP}^2. \)

Set \( U_1 = \left\{ [x', x^2, x^3] \mid x' \neq 0 \right\}. \)

\[ = \text{set of lines not in } x^2 x^3 \text{-plane} \]

Define \( \phi_1 : U_1 \to \mathbb{R}^2 \text{ by } \)
by \[ \phi_1(\mathbf{x}') = \left( \frac{x^2}{x^1}, \frac{x^3}{x^1} \right) \].

This is well defined since
\[ \phi_1(\lambda x', \lambda x^2, \lambda x^3) = \left( \frac{\lambda x^2}{\lambda x^1}, \frac{\lambda x^3}{\lambda x^1} \right) = \left( \frac{x^2}{x^1}, \frac{x^3}{x^1} \right) \].

Check \((\phi_1, U_1)\) is a chart.

\cdot \phi_1(U_1) = \mathbb{R}^2 \text{ which is open.}

\cdot \phi_1(\mathbf{x}') = \phi_1(\mathbf{y}', \mathbf{y}^2, \mathbf{y}^3)

\Rightarrow \left( \frac{x^2}{x^1}, \frac{x^3}{x^1} \right) = \left( \frac{y^2}{y^1}, \frac{y^3}{y^1} \right)

\Rightarrow x^2 = \frac{x^1}{y^1} \, y^2 \quad \text{and} \quad x^3 = \frac{x^1}{y^1} \, y^3

\Rightarrow (x^1, x^2, x^3) = \frac{x^1}{y^1} (y^1, y^2, y^3)

\Rightarrow x \sim y.

Note \(\phi_1^{-1}(u,v) = [1, u, v]\)

Define \((U_2, \phi_2)\) and \((U_3, \phi_3)\) analogously.
Claim \[ a = \left\{ (u_i, \phi_i) \right\}_{i=1,2,3} \] is a smooth atlas on \( \mathbb{RP}^2 \).

**Proof** \( [x^1, x^2, x^3] \in \mathbb{RP}^2 \Rightarrow x^i \neq 0 \) for some \( i \)

\[ \Rightarrow [x^1, x^2, x^3] \in U_i \]

Consider compatibility of \( \phi_1 \) and \( \phi_2 \).

\[ \phi_2 \left( U_1 \cap U_2 \right) = \phi_2 \left( \left\{ [x^1, x^2, x^3] \mid x^1 \neq 0, x^2 \neq 0 \right\} \right) \]

\[ = \left\{ \left( \frac{x^1}{x_1}, \frac{x^2}{x_1} \right) \mid x^1 \neq 0, x^2 \neq 0 \right\} \]

\[ = \left\{ (u, v) \mid u \neq 0 \right\} \]

Exercise: this is open.

Now \( \phi_1 \circ \phi_2^{-1} (u, v) = \phi_1 \left( [u, 1, v] \right) \)
\[ = \left( \frac{1}{u}, \frac{v}{u} \right) \]

This is smooth on its domain \((u \neq 0)\).