**Def** (provisional)

A **smooth manifold** is a set \( M \) equipped with a **smooth atlas** \( \mathcal{A} = \{ (U_\alpha, \phi_\alpha) \}_{\alpha \in A} \).

Let's deal with the "provisional"

**Def** a chart \( (U, \phi) \) is compatible with \( \mathcal{A} \) if

it is compatible with each \( (U_\alpha, \phi_\alpha) \in \mathcal{A} \).

**Exercise** If \( (U, \phi) \) is compatible with \( \mathcal{A} \), then

\( \mathcal{A} \cup \{ (U, \phi) \} \) is an atlas on \( M \).

**Basic Problem** It is easy to find \( (U, \phi) \) compatible with \( \mathcal{A} \) and we don't want to view \( (M, \mathcal{A}) \) and \( (M, \mathcal{A} \cup \{ (U, \phi) \}) \) as different smooth manifolds.

**Method 1: Restriction**

Choose \( (U_\alpha, \phi_\alpha) \in \mathcal{A} \) and \( V \subset \) open \( \phi_\alpha(U_\alpha) \)
Exercise \((\phi^{-1}_x(V), \phi_x)\) is a chart compatible with \(\alpha\).

Method Composition

Let \(\Psi: \mathbb{R}^n \to \mathbb{R}^n\) be smooth with smooth inverse.

If \(A\) is an invertible matrix and \(c \in \mathbb{R}^n\),

the map \(\Psi(x) = Ax + c\) is smooth as is its inverse \(\Psi^{-1}(y) = A^{-1}(y - c)\).

Exercise. For \((U_x, \phi_x) \in \alpha\), \((U_x, \Psi \circ \phi_x)\) is a chart compatible with \(\alpha\).
Let's deal with the Basic Problem.

**Def** two atlases $\mathcal{A}$ and $\mathcal{A}'$ on $M$ are compatible ($\mathcal{A} \sim \mathcal{A}'$) if each chart of $\mathcal{A}$ is compatible with each chart of $\mathcal{A}'$.

**Def** (??) A smooth manifold is a set $M$ equipped with an equivalence class of smooth atlases, $[\mathcal{A}]$.

Now $(M, [\mathcal{A}]) = (M, [\mathcal{A} \cup (u, \phi)])$

There is still a subtle (topological) problem.

Given an atlas $\mathcal{A}$ for $M$ add to it all charts compatible with $\mathcal{A}$ to get $\text{max}(\mathcal{A})$.

- $\text{max}(\mathcal{A})$ is an atlas on $M$
- $[\text{max}(\mathcal{A})] \sim [\mathcal{A}]$.
- Every chart compatible with $\text{max}(\mathcal{A})$ is in $\text{max}(\mathcal{A})$.
- $\mathcal{A} \sim \mathcal{A}' \iff \text{max}(\mathcal{A}) = \text{max}(\mathcal{A}')$. 
Topology from an atlas.

Let \( A \) be an atlas on \( M \). \((M, [A])\) is smooth if \( W \subset M \) is open if for each \( p \in W \) there is a \((U, \phi) \in \text{max}(A)\) such that \( p \in U \subset W \).

This collection of open sets of \( M \) defines a topology on \( M \).

**Def** A smooth manifold is a pair \((M, [A])\) such that the (induced) topology on \( M \) is Hausdorff and has a countable base.

**Hausdorff**: For \( p \neq q \in M \) there are open sets \( U, V \subset M \) s.t. \( p \in U \), \( q \in V \) and \( U \cap V = \emptyset \).

**Example** \( M = \mathbb{R} \setminus \{0\} \cup \{p_1, p_2\} \)

\[ \xrightarrow{-} \{p_1, p_2\} \]
$U_1 = M \setminus \{ p_2 \} \quad U_2 = M \setminus \{ p_3 \}$

$\Phi_1 : U_1 \to \mathbb{R} \quad \Phi_2 : U_2 \to \mathbb{R}$

$x \mapsto x \quad x \mapsto x$

$p_1 \mapsto 0 \quad p_2 \mapsto 0$

$\{ (U_1, \Phi_1), (U_2, \Phi_2) \}$ is a smooth atlas on $M$

The topology on $M$ is not Hausdorff.

Let $(M, \mathcal{E}A)$ be a smooth manifold of dimension $n$.

Let $\mathbb{C}^\infty(M) = \{ \text{smooth functions on } M \}$

$\mathbb{C}^\infty(M)$ is a vector space.

$(f + g)(p) = f(p) + g(p)$

$(cf)(p) = c f(p)$

Let $f : M \to \mathbb{R}$ be smooth.

Q. What is the derivative of $f$ at $p \in M$?

A. It is a linear map from $T_p M$ to $\mathbb{R}$.

Q. What is $T_p M$?
By name, $T_pM$ is the tangent space to $M$ at $p$.

- $T_pM$ is a vector space of dimension $n$.
- $T_pM$ is a collection of linear maps from $C^\infty(M)$ to $\mathbb{R}$.

- Each chart $(U, \phi)$ with $p \in U$ determines a basis for $T_pM$.

Let's look at things in Example 0.

$M = \bigcup_{\text{open}} U \subset \mathbb{R}^n \quad \mathcal{A} = \{ (U, Id_U) \}$.

- A tangent vector to $M$ at $x$ is a vector $v$ with base at $x$.
- The space of all such vectors is a copy of $\mathbb{R}^n$ with origin at $x$. Hence a vector space of dimension $n$. 
Each $v$ determines a map

$$D_v : C^\infty(M) \longrightarrow \mathbb{R}$$

$$f \longmapsto \frac{df}{dt} \bigg|_{t=v}$$

This map is linear.

$$D_v (f + cg) = \frac{d}{dt} \bigg|_{t=0} (f + cg)(x + tv)$$

$$= \frac{d}{dt} \bigg|_{t=0} \left( f(x+tv) + c \cdot g(x+tv) \right)$$

$$= D_v (f) + c \cdot D_v (g)$$

This map satisfies the product rule

$$D_v (fg) = D_v f \cdot g(x) + f(x) \cdot D_v g$$

**Fact.** Any linear map satisfying this rule is equal to $D_v$ for some $v$. 
So \[ v \leftrightarrow D_v \]

**Def:** For a smooth manifold \((M, [\mathcal{A}])\), the tangent space to \(M\) at \(p\) is

\[ T_p M = \left\{ L : C^\infty(M) \to \mathbb{R} \mid L \text{ is linear and} \quad L(fg) = L(f)g(p) + f(p)L(g) \right\} \]