\[ X : M \rightarrow TM \text{ s.t. } \pi \circ X = \text{Id}_M \]
\[ X_0(p) = (p, V(p)) \]

If \( M \) is compact or \( \text{supp} X \) is compact, the flow \( \Phi_t \) of \( X \) is global.

**Example** \( TS^1 \)

- We constructed a nonvanishing vector field on \( S^1 \)

\[ X(x) = (x, \dot{x}(t)) \quad \text{where} \quad \dot{x}(t) = x. \]

- Another version of the same vector field is

\[ X(x) = \left( (x^1, x^2), -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} \right) \]

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{vector_field_diagram}
\end{array}
\]

This yielded a diffeomorphism

\[ F : S^1 \times \mathbb{R} \rightarrow TS^1 \]

\[ (x, c) \mapsto (x, c \frac{\partial}{\partial \theta}) \]
Fact. \( TM \cong M \times \mathbb{R}^n \) if and only if there are \( n \) vector fields \( X^1, \ldots, X^n \) on \( M \) such that at each \( p \), the collection \( \{ V^1(p), \ldots, V^n(p) \} \) spans \( T_p M \) where \( X^i(p) = (p, V^i(p)) \).

Example. \( TS^2 \)

Again we can picture this ...

![Diagram of a sphere with vector fields]

Then. Every vector field \( X \) on \( S^2 \) vanishes at some pt. i.e. \( \exists p \in S^2 \) such that \( X(p) = (p, \overline{0} \in T_p S^2) \).

Corollary. \( TS^2 \neq S^2 \times \mathbb{R}^2 \).
OK, let's take stock.

\[ R^m \rightarrow M \rightarrow \mathcal{F} \rightarrow \mathcal{F} : M ightarrow N \]

\[ (\frac{\partial \Phi}{\partial x^i} (x)) \rightarrow F^* : T_p M \rightarrow T_{f(p)} N \]

We can do optimization on \( M \) and solve IVPs.

Next we would like to integrate!

The things we integrate will be special tensor fields.

Aside: Dual Spaces

Let \( V \) be a finite dimensional vector space (over \( \mathbb{R} \)).

\[ V^* = \{ \alpha : V \rightarrow \mathbb{R} \mid \alpha \text{ is linear} \} \]

Prop. \( V^* \) is a vector space w.r.t. the operation

\[ (\alpha + c\beta)(V) = \alpha(V) + c(\beta(V)) \]  

Moreover, \( \dim(V^*) = \dim(V) \).
Let \( \{e_1, \ldots, e_n\} \) be a basis for \( V \).

Define \( \sigma^i \in V^* \) by \( \sigma^i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \).

So, \( \sigma^i(v) = \sigma^i(a_1e_1 + \cdots + a_ne_n) \)

\[= a_1 \sigma^i(e_1) + \cdots + a_n \sigma^i(e_n)\]

\[= a_i \]

**Exercise** Prove that \( \{\sigma^1, \ldots, \sigma^n\} \) is a basis for \( V^* \).

\( \{\sigma^i\} \) is the **basis dual to** \( \{e_j\} \).

**Aside** The dot product on \( V = \mathbb{R}^n \) defines an isomorphism \( \mathbb{R}^n \rightarrow (\mathbb{R}^n)^* \) where \( x \mapsto \sigma_x \)

\[\sigma_x(y) = x \cdot y.\]

Consider the vector space \( T_pM \)

\[(T_pM)^* = T_p^*M\] the cotangent space to \( M \) at \( p \).
• Each $f \in \mathcal{C}^\infty(M)$ determines an element $df(p) \in T_pM$ (for each $p \in M$) via the formula

$$df(p)(X) = X(f),$$

$$df(p) : T_pM \to \mathbb{R}$$ is the differential of $f$ at $p$.

**Example 0**

On $M = \mathbb{R}^n$ we have the functions

$$x^i : \mathbb{R}^n \to \mathbb{R}$$ (we call this $T^n_i$ previously)

$$x \mapsto x^i$$

$$Jx^i(x) : T_x \mathbb{R}^n \to \mathbb{R}$$

**Claim**

$\{dx^i(x)\}$ is the basis of $T^*_x \mathbb{R}^n$ dual to $\{\frac{\partial}{\partial x^i}\}^*_x$.

$$dx^i(x) \left( \frac{\partial}{\partial x^j} \right)_x = \frac{\partial^2}{\partial x^i \partial x^j} \bigg|_x (x^j) = \delta^i_j$$
Example 1 \: \text{let } (U, \phi) \text{ be a chart.}

Consider \( x^i_u : U \rightarrow \mathbb{R} \)

\[ p \mapsto x^i(\phi(p)) \]

\( dx^i_u(p) \in T^*_p M \) is defined by

\[ dx^i_u(p)(V) = \nabla(V, x^i_u) \]

\( \{ dx^i_u(p) \} \) is basis dual to \( \{ \frac{\partial}{\partial x^k_u} \big|_p \} \).

\[ dx^i_u(p) \left( \frac{\partial}{\partial x^k_u} \big|_p \right) = \frac{2}{dx^i_u} \big|_p \left( x^i_u \right) \]

\[ = \frac{2}{dx^i} \big|_\phi(\gamma) \left( x^i \circ \phi \circ \phi^{-1} \right) \]

\[ = \frac{2}{dx^i} \big|_\phi(\gamma) \left( x^i \right) \]

\[ = \delta^i_j \]
Change of coordinates for differentials

\[ d\bar{x}^i_v = \sum_j \frac{\partial x^i_v}{\partial x^j_u} d\bar{x}^j_u \]

Claim: Given \( f \in C^\infty(M) \), \((U, \Phi)\) a chart and \( p \in U \)

\[ df(p) = \sum_i \frac{\partial f}{\partial x^i_u}(p) d\bar{x}^i_u(p) \]

Cor: \( df(p) = 0 \in T_p^* M \iff p \) is a critical point of \( f \)

In Calculus one calls the differential of \( f \)

\[ df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n \]

but doesn't say where it lives.

Def: The cotangent bundle of \( M \) is the set

\[ T^* M = \left\{ (q, \omega) \mid q \in M, \omega \in T_q^* M \right\} \]

Then \( T^* M \) inherits from \( M \) the structure of a smooth manifold of dimension \( 2 \dim(M) \).

The
map \ \pi: T^*M \rightarrow M \ \text{is smooth.} \hspace{2cm} \hspace{2cm} \hspace{2cm} \hspace{2cm} \\
(\eta, \alpha) \mapsto \hat{\eta}