Integrating differential forms

**Step 0** \( U = [0,1]^n \subset \mathbb{R}^n \)

\[ \alpha = f(x) \, dx^1 \wedge \ldots \wedge dx^n \]

\[ \int_U \alpha = \int_{\square} \cdots \int_{\square} f(x_1', \ldots, x_n') \, dx_1' \wedge \ldots \wedge dx_n' . \]

A) **Pullback.** \( \mathcal{F} : M \rightarrow N \)

\( \alpha \in \Lambda^k(N) \)

\[ \Lambda^k(N) \ni (\mathcal{F}^\ast \alpha)(V_1, \ldots, V_k) = \alpha((\mathcal{F}^\ast \alpha)(\mathcal{F}_* V_1, \ldots, \mathcal{F}_* V_k)) \]

**Ex** \( \mathcal{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \)

\((x^1, x^2) \mapsto (x^1 - x^2, x^2 - x^1, (x^1)^2)\)

\(\alpha(y) = \sqrt{y^1} \, dy^1 \wedge dy^3 \in \Lambda^2(\mathbb{R}^3)\)

\((\mathcal{F}^\ast \alpha)(\alpha) = \sqrt{x^1 - x^2} \, d(x^1 - x^2) \wedge d((x^1)^3)\)

\[ = \sqrt{x^1 - x^2} \left[ (dx^1 - dx^2) \wedge (3(x^1)^2 \, dx^1) \right] \]

\[ = -\sqrt{x^1 - x^2} \, 3(x^1)^2 \, dx^2 \wedge dx^1 \]
\[ \omega = \sqrt{x^1 - x^2} \, (3A^1)^2 \, dx^1 \wedge dx^2. \]

**Ex**  \( \gamma_i : (0, 1) \rightarrow \mathbb{R}^n \)

\[ t \mapsto (\gamma_1(t), \ldots, \gamma_n(t)) \]

\[ \kappa(\omega) = \sum a_i \, \omega_i \, dx^i \]

\[ (\gamma^\ast \kappa)(t) = \sum a_i (\gamma_i(t)) \, d(\gamma_i(t)) \]

\[ = \sum a_i (\gamma_i(t)) \, \frac{d\gamma_i(t)}{dt} \, dt \]

\[ = \left[ \sum a_i (\gamma_i(t)) \, dx^i \right] \left( \sum \frac{d\gamma_i(t)}{dt} \frac{\partial}{\partial x^i} \right) \, dt \]

\[ = \kappa(\gamma(t)) \, (\dot{\gamma}(t)) \, dt. \]

So

\[ \int_{\gamma} \kappa = \int_{0}^{1} \kappa(\gamma(t)) \, (\dot{\gamma}(t)) \, dt = \int_{a}^{b} \gamma^\ast(\kappa) \]

**Step 1**  
Given \( \omega \in \Lambda^k(M) \) and a smooth map

\[ \gamma : [0, 1]^k \rightarrow M \]  

(singular \( k \)-cube)

\[ \int_{\gamma} \omega = \int_{[0, 1]^k} \gamma^\ast \omega \quad \text{as defined in Step 0}. \]
Now suppose $UCM$ and $U = \gamma([0,1]^n)$ for some $\gamma: [0,1]^n \to M$ which is smooth and 1-1.

Q. Does it make sense to define $\int_U \omega = \int_{\partial U}$?

A. NO. There is a sign problem.

Example: $M = S^1$, $U = \{x \in S^1 \mid x^2 \geq 0\}$

\[ \gamma_1: [0,1] \to S^1 \]
\[ s \mapsto (\cos(\pi s), \sin(\pi s)) \]

\[ \gamma_2: [0,1] \to S^1 \]
\[ s \mapsto (\cos(\pi(1-s)), \sin(\pi(1-s))) \]

$\gamma_1([0,1]) = U = \gamma_2([0,1])$

Exercise: For any 1-form $\alpha$ on $S^1$, $\int_{\gamma_1} \alpha = -\int_{\gamma_2} \alpha$. 
We need a way to consistently choose b/w signs.

B) Orientations

Let $\alpha = \{e_1, \ldots, e_n\}$ and $\beta = \{f_1, \ldots, f_n\}$ be basis for $V$.

$[\text{Id}_V]_\alpha^\beta$ is an invertible matrix. 

$$(( [\text{Id}_V]_\alpha^\beta )^{-1} = [\text{Id}_V]_\beta^\alpha)$$

Def: $\alpha \sim \beta$ iff $\det ([\text{Id}_V]_\alpha^\beta) > 0$

Ex: $V = \mathbb{R}^2$ 

$\alpha = \{ (1,0) , (0,1) \}$

$\beta = \{ (0,1) , (1,0) \}$

$\gamma = \{ (1,1) , (1,-1) \}$

$[\text{Id}]^\alpha_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so $\alpha \not\sim \beta$.

$[\text{Id}]^\gamma_\alpha = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ so $\alpha \not\sim \gamma$.

$[\text{Id}]^\gamma_\beta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ so $\beta \sim \gamma$. 
Fact. For any \( V \) there are exactly 2 equivalence classes of bases \( \{+, -\} \).

Def. An orientation of \( V \) is a choice of one equivalence class of bases \( (+) \).

Let \((U, \phi)\) and \((V, \psi)\) be charts on \( M \).

Def. \((U, \phi)\) and \((V, \psi)\) are orientation compatible if either \( U \cap V = \emptyset \) or if:

\[
\det \left( \begin{bmatrix} (\psi \circ \phi^{-1})_* & \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial x} \end{bmatrix} \right) > 0
\]

for all \( x \in \phi(U \cap V) \).

Ex. \( M = S^1 \) \((U_1^+, \phi_1^+) \) \((U_2^+, \phi_2^+) \)

\[
\begin{array}{c}
\end{array}
\]
\[ \phi_2^+ \cdot (\phi_1^+)^{-1} (x^i) = \phi_2^+ \left( \sqrt{1-(x^i)^2}, x^i \right) \]
\[ = \frac{2}{\partial x^i} \left( \sqrt{1-(x^i)^2} \right) \]
\[ = \frac{-x^i}{\sqrt{1-(x^i)^2}} < 0 \]

So not orientation compatible.

Ex: Change \( \phi_2^+ \) to \( \tilde{\phi}_2^+ (x^i, x^s) = -x^i \).

Then \( (U_1^+, \phi_1^+) \) and \( (U_2^+, \tilde{\phi}_2^+) \) are orientation compatible.

Def: \( M \) is orientable if it admits an atlas \( \mathcal{A} = \{ (U_{\alpha}, \phi_{\alpha}) \}_{\alpha \in \mathcal{A}} \) such that any two charts in \( \mathcal{A} \) are orientation compatible.

Such an atlas is said to be orienting.
Rank  Not all manifolds are orientable!

ex $\mathbb{RP}^2$ is not orientable

**Def** If $A$ and $B$ be orienting atlases for $M$, $A \sim B$ if every pair of charts $(U, \phi) \in A$

and $(V, \psi) \in B$ are orientation compatible.

**Fact** If $M$ is orientable it has exactly two equivalence classes of orienting atlases.

**Def** An orientation on an orientable manifold is a choice of one of these equivalence classes.

**Def** $M$ oriented. $(U, \phi)$ is positively oriented if it belongs to an oriented atlas in the chosen equivalence class.
\[ M = \mathbb{R}^n \]

\[ A = \{(\mathbb{R}^n, \text{Id}_{\mathbb{R}^n})\} \text{ is an orienting atlas.} \]

\[ B = \{(\mathbb{R}^n, -\text{Id}_{\mathbb{R}^n})\} \text{ is an orienting atlas.} \]

\[ A \sim B \iff n \text{ is even} \]

\[ \text{det} \left[ (-\text{Id}_{\mathbb{R}^n}) \cdot (\text{Id}_{\mathbb{R}^n})^{-1} \right] = \text{det} (-\mathbb{I}_n) = (-1)^n. \]

Ref: Suppose \( M \) is oriented. \( \gamma: [0,1]^n \to M \) is positively oriented if \( \text{det} \left[ (\phi \circ \gamma)_* \right] \) for any positively oriented chart on \( (U,d) \) in \( M \) (with \( U \cap \gamma([0,1]^n) \neq \emptyset \)).