Integrating differential forms

**Step 1** \( U = [0,1]^n \subset \mathbb{R}^n \)
\( \alpha = \int f(x) \, dx^1 \wedge \cdots \wedge dx^n \)
\( \int \alpha = \int_{\gamma} \int \cdots \int f(x_1', \ldots, x_n') \, dx_1' \cdots dx_n' \).  

**Step 2** \( \gamma: [0,1]^k \to M \), \( \alpha \in \Lambda^k(M) \)
\( \int \alpha = \int_{[0,1]^k} \gamma^* \alpha \)

Given \( \gamma: [0,1]^n \to M \) 1-1 and \( \alpha \in \Lambda^k(M) \) one would hope that \( \int \alpha = \int_{[0,1]^n} \gamma^* \alpha \) is a good definition.

It is not. There is a sign problem.

\[ \begin{array}{c}
\text{M} \\
\gamma \\
\tilde{\gamma} \\
\end{array} \]
\[ \int_Y \omega = - \int_Y \omega \]

Orientability

- Two bases, \( \alpha, \beta \) of \( V \) are equivalent if \( \det [\text{Id}]^\beta_\alpha > 0 \).

- A map \( T : (V, \mathcal{E}) \to (W, \mathcal{F}) \) is orientation preserving if \( \det [T]^\beta_\alpha > 0 \).

- Two charts \((U, \phi)\) and \((V, \psi)\) of \( M \) are orientation compatible if

\[
\det \left[ (\psi \circ \phi^{-1}) \right] > 0 \quad \text{(for all } x \in \phi(U \cap V) \text{)}
\]

Example:

\[
M = S^1 \quad (U_1^+, \phi_1^+) \quad (U_2^+, \phi_2^+)
\]

\[
\phi_2^+ \circ (\phi_1^+)^{-1} (x^1) = \phi_2^+ \left( (1-\cos x^1), x^1 \right) = \sqrt{1-\cos^2}
\]

\[
\left[ (\phi_2^+ \circ (\phi_1^+)^{-1}) \right] = \frac{2}{\cos} \left( \sqrt{1-\cos^2} \right)
\]
\[
\frac{-x^1}{\sqrt{1-|x'|^2}} < 0 \quad \forall x' \in \phi_1^+(U_+^1 \cap U_+^2) = (0,1)
\]

So not orientation compatible.

Ex Change $\phi_2^+$ to $\tilde{\phi}_2^+(x',x^2) = -x^1$.

Then $(U_1^+, \tilde{\phi}_1^+)$ and $(U_2^+, \tilde{\phi}_2^+)$ are orientation compatible.

Def $M$ is orientable if it admits an atlas

\[ A = \bigcup \left( U_n, \phi_n \right) \] such that any two charts in $A$ are orientation compatible.

Such an atlas is said to be orienting.

Remk Not all manifolds are orientable!

ex $\mathbb{RP}^2$ is not orientable.

Def Let $A$ and $B$ be orienting atlases for $M$.

$A \sim B$ if every pair of charts $(U, \phi) \in A$

and $(V, \psi) \in B$ are orientation compatible.
Fact If $M$ is orientable it has exactly two equivalence classes of orienting atlases.

Def An orientation on an orientable manifold is a choice of one of these equivalence classes.

An oriented manifold is a pair $(M, [\mathcal{A}])$.

Ex $M = \mathbb{R}^n$
- $A = \{ (\mathbb{R}^n, Id_{\mathbb{R}^n}) \}$ is an orienting atlas
- $B = \{ (\mathbb{R}^n, -Id_{\mathbb{R}^n}) \}$ is an orienting atlas

$A \sim B \iff n$ is even

i.e. $\det \left( (-Id_{\mathbb{R}^n}) \cdot (Id_{\mathbb{R}^n})^{-\frac{1}{n}} \right) = \det (-I_n) = (-1)^n$.

Def A chart $(U, \phi)$ of $(M, [\mathcal{A}])$ is positively oriented if it belongs to an oriented atlas in the chosen equivalence class.
Ref² \( \gamma : [0,1]^n \to (M, [\mathcal{A}]) \) is positively oriented if \( \det \left( (\phi \circ \gamma)_* \right) > 0 \) for any positively oriented chart on \((U, \phi)\) on \(M\) (and any \( x \in [0,1]^n \)).

**Step 2**

Let \( M \) be oriented. Suppose \( \gamma : [0,1]^n \to M \) is 1-1 and positively oriented. Then for \( U = \gamma([0,1]^n) \) we define

\[
\int_x^y = \int_x^y \quad \text{for any } x, y \in \Lambda^*(M),
\]

**Remark** \( \int_x^y \) makes sense for \( \gamma : [0,1]^n \to \mathbb{R}P^n \)

\( \int_x^y \) does not.

Assume \( M \) is oriented and \( x \in \Lambda^*(M) \)

We want to define \( \int_M^x \) by dividing \( M \) up into pieces as in Step 2.
Assume $M$ is compact.

Then $M$ admits a finite open cover $\bigcup_{i=1}^{N} V_i$. This means:

- Each $V_i \subset M$ is open.
- $\bigcup_{i=1}^{N} V_i = M$.

**Prop.** $M$ (compact) admits a finite open cover such that $V_i \subset \overline{\delta_i}([0,1]^n)$ for some $\delta_i : [0,1]^n \to M$ 1-1 and orientation preserving.

What about \[ \int_M \alpha = \sum_{i=1}^{N} \int_{\overline{\delta_i}([0,1]^n)} \alpha \] ?

**Problem:** overlapping contributions

Define a partition of unity subordinate to $\bigcup V_i \beta_{i=1}^{N}$.
a collection of smooth functions \( \{ f_i : M \to \mathbb{R}^3 \}_{i=1}^N \)

s.t.

i) \( f_i(p) \geq 0 \quad \forall \ i, \ p \)

ii) \( f_i(p) = 0 \quad \forall \ p \notin V_i \)

iii) \( \sum_{i=1}^N f_i(p) = 1 \quad \forall \ p \)

i.e.

\[
\begin{array}{c}
V_i \\
\bigcap \bigcup \bigcup \\
\bigcap \bigcup V_j
\end{array}
\]

Then there is a partition of unity subordinate to any finite (in fact countable) open cover of \( M \).

Given \( M \) oriented and \( \alpha \in \Lambda^*(M) \)

- choose \( \sum V_i \beta_i^N \) such that \( V_i < \delta_1(0,1)^n \) as in Prop.
- choose a partition of unity subordinate to \( \sum V_i \beta_i^N \) as in Thm.

Def \( \int_M \alpha = \int_M 1 \wedge \alpha \)
\[
\int \left( \sum_{i=1}^{N} f_i \right) \chi_M = \sum_{i=1}^{N} \int_M f_i \chi_M
\]

since \( f_i = 0 \) outside \( V_i \subset \delta_i([0,1])^n \).

\[
= \sum_{i=1}^{N} \int_{[0,1]^n} \delta_i^*(f_i \chi_M) \quad \text{by Step 2.}
\]