Then 1 \( \langle X, Y \rangle = [X, Y] \)

\[
\lim_{t \to 0} \frac{1}{t} \left( (\phi_t^X)(Y(\phi_t^X) - Y) \right) = XY - YY
\]

PF: It suffices to prove that \( \langle X, Y \rangle = [X, Y] \)

near general p.m.

Then 2 (Flow box theorem)

If \( X(p) \neq 0 \) then there is a chart \( (U, \phi) \) near \( p \)

such that \( X(x) = \frac{\partial f}{\partial y} \bigg|_x \)

\[
\begin{array}{c}
\text{idea} \\
X(\psi^x) = \int_{(t, \psi^x) \in U} \frac{\partial f}{\partial y} \\
y = \int_0^\psi \frac{\partial f}{\partial y} ds \\
X(y) = \int_0^\psi \frac{\partial f}{\partial y} ds = \int_0^\psi \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial y}.
\end{array}
\]

Back to Then 1.

- If \( X(p) = 0 \) then \([X,Y](p) = 0\) and

\[
\langle X, Y \rangle (p) = \lim_{t \to 0} \frac{1}{t} \left( Y(p) - (\phi_t^X)^* (Y \circ \phi_t^X(p)) \right)
\]
\[ = 0 \]

If \( X \not= 0 \) then in new box coordinates

\[ X(x^i) = \frac{\partial}{\partial x^i}, \quad Y(x^i) = \sum X^i \frac{\partial}{\partial x^i} \]

\[ [X, Y] = \sum_j \left( \sum_i X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \]

\[ = \sum_j \frac{\partial Y^j}{\partial x^i} \]

\[ \phi_t^x (x^1, \ldots, x^n) = (x^1 + t, x^2, \ldots, x^n) \]

\[ [\left( \phi_t^x \right)_*] = I_n \]

\[ (\phi_{-t}^x)_* \gamma (\phi_t^x (x)) = (\phi_{-t}^x)_* \left( \sum_i \gamma^i (x^1, x^2, \ldots, x^n) \frac{\partial}{\partial x^i} \right) \]

\[ = \sum_i \gamma^i (x^1, x^2, \ldots, x^n) \]

\[ \mathcal{L}_X \gamma (x) = \lim_{t \to 0} \frac{1}{t} \left( \sum_i \left( \gamma^i (x^1 + t, x^2, \ldots, x^n) - \gamma^i (x) \right) \frac{\partial}{\partial x^i} \right) \]
\[ \sum_i \frac{\partial Y^i}{\partial x^i} \]

For a \( k \)-form \( \omega \in \Lambda^k(M) \) we can also define

\[ \mathcal{L}_X \omega(p) = \frac{d}{dt} \bigg|_{t=0} ((\phi_t^* \omega)(p)) \]

Thm 3: Cartan's Magic formula.

\[ \mathcal{L}_X \omega = d(\omega(X, \cdot)) + d\omega(X, \cdot) \]

Exercise: Prove this using idea above.

Connections on \( M \).

Unlike \( Xf = \sum_i X^i \frac{\partial f}{\partial x^i} \), \( \mathcal{L}_X Y = \sum (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j} \) depends on the values of \( X \) in a nbhd of \( p \).

Q. Can we define a directional derivative of \( Y \) w.r.t. \( X \) at \( p \) which only depends on \( X(p) \)?

A connection at \( p \in M \) is a bilinear map

\[ T_p M \times \{ \text{vector fields} \} \rightarrow T_p M \]
\[(X(p), Y) \rightarrow \nabla_{X(p)} Y\]

s.t. \[\nabla_{X(p)} fY = f(\nabla_{X(p)} Y) + X(f) Y,\]

A connection on \(M\) is a map which assigns to each \(p\) a connection \(\nabla(p)\) at \(p\) such that

1) For vector fields \(X, Y\) on \(M\)
\[\nabla_X Y(p) = \nabla_{\gamma(p)} Y\] is a smooth vector field.

2) \((X, Y) \rightarrow \nabla_X Y\) is bilinear

3) \[\nabla_{fX} Y = f \nabla_X Y\]

4) \[\nabla_X (fY) = f(\nabla_X Y) + X(f) Y\]

Let's look at such a \(\nabla\) must look like in local coordinates
\[\nabla_X Y = \nabla_X \sum y^i \frac{\partial}{\partial x^i}\]

\[= \sum_{i,j} \left( \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial x^i} + \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \right)\]
\[ \nabla \text{ is determined by } \nabla \frac{\partial}{\partial x_i} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k} \]

\[ \nabla \text{ is determined by } n^3 \text{ functions } \Gamma^k_{ij} \text{ (Christoffel symbols)} \]

There are LOTS of connections on \( M \).

For \( \Gamma^k_{ij} = 0 \Rightarrow \nabla \frac{\partial}{\partial x_i} Y = \nabla \frac{\partial}{\partial x_i} (\sum Y^j \frac{\partial}{\partial x_j}) = \sum_j \frac{\partial Y^j}{\partial x_i} \frac{\partial}{\partial x_j} \]