A connection on $M$ is a bilinear map

$\{ \text{vector fields} \} \times \{ \text{vector fields} \} \longrightarrow \{ \text{vector fields} \}$

$(X, Y) \longrightarrow \nabla_X Y$

\[ \nabla_{fX} Y = f \left( \nabla_X Y \right) \]

\[ \nabla_X (fY) = f \left( \nabla_X Y \right) + (Xf)Y \]

Locally $\nabla$ is determined by

\[ \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x_k} \text{ Christoffel Symbols of } \nabla. \]

Example \( \mathbb{R}^2 \).

The standard ($\Gamma^k_{ij} = 0$) connection on $\mathbb{R}^2$

\[ \nabla_{\frac{\partial}{\partial x_i}} Y = \sum_j \frac{\partial Y^j}{\partial x_i} \frac{\partial}{\partial x^j} \]

Note: ANY collection of $n^2$ numbers $\Gamma_{ij}^k$ determines a connection on $\mathbb{R}^n$.

Recall: We also have lots of metrics on $M$.

These two choices can be made in concert.
Thus for every metric $g$ on $M$ exists a connection $\nabla$ with the following additional properties:

1. $\nabla_X Y - \nabla_Y X = [X,Y]$ (Torsion Free)

2. $X(\langle Y,Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$ (covariant derivative is compatible with $g$)

$(\nabla$ encodes $g$-Leibnitz Rule$)$

This unique $\nabla$ corresponding to $g$ is called its Riemannian connection.

In local coordinates $g(x) = \sum_{ij} g_{ij}(x) \, dx^i \otimes dx^j$

and $\Gamma^k_{ij} = \frac{1}{2} \sum_x g_{kk} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} \right)$

where $(g_{ij})^{-1} = (g^{ij})$

OK, what can we do with $\nabla$ and why is it called a connection?
Parallel Translation

Given $p, q \in M$, want a isomorphism from $T_p M \to T_q M$. Let $\gamma : [0, 1] \to M$ be smooth s.t. $\gamma(0) = p$, $\gamma(1) = q$.

A vector field along $\gamma$ is a map $V : [0, 1] \to TM$ s.t. $V(t) \in T_{\gamma(t)} M$.

**ex 1** $\dot{\gamma}(0)$ is a v.f. along $\gamma$.

**ex 2** A vector field $X$ on $M$ determines one along $\gamma$, $X_{\gamma(t)}$.

**ex 3**
Not every $v(t)$ along $\gamma$ is a restriction as in $\mathbb{R}^2$!

A connection $\nabla$ on $M$ defines a unique map

$$\{\text{Vector fields along } \gamma \} \rightarrow \{\text{Vector fields along } \gamma \}$$

$V \rightarrow \nabla_\gamma V$

5.1 a) $\nabla_\gamma (V + W) = \nabla_\gamma V + \nabla_\gamma W$

b) $\nabla_\gamma (fV) = f \nabla_\gamma V + \frac{df}{dt} V$

c) If $V = X(t(\gamma))$ then $\nabla_\gamma V = \nabla_\gamma X$.

In local coordinates $V(t) = v^i(1) \frac{\partial}{\partial x^i} + \ldots + v^n(1) \frac{\partial}{\partial x^n}$

and

$$\left( \nabla_\gamma V \right)(t) = \sum_k \left( \frac{d}{dt} v^k(t) + \sum_{i,j} \Gamma^k_{ij} \dot{g}^i(t) v^j(t) \right) \frac{\partial}{\partial x^k}$$

Recall A v.f. $V$ along $\gamma$ is parallel if $\nabla_\gamma V = 0$. 
Then, given $\gamma : [0, 1] \to M$ smooth at $V_0$ and $V_0$, there is a unique parallel $\gamma^t$ to $V$ along $\gamma$ such that $V(0) = V_0$.

**Proof**

Apply the Existence and Uniqueness Theorem to

$$V^t_{C(1)} + \sum_{i,j} \Gamma^k_{ij}(V(t)) \gamma^t_i(V(t)) V^t_j(V(t)) = 0$$

This are linear equation first order ODE's.

Then, the map $T_pM \to T_qM$ is an isomorphism:

$$V_0 \to V(1)$$

(it depends on $\nabla$).

Hence "connection"
Fix a metric $g$ on $M$. Let $\nabla$ be the unique Riemannian connection for $g$.

**Definition** $\gamma : (a, b) \to M$ is a geodesic for $g$ if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ ("acceleration" = 0).

In local coordinates

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

$$\Rightarrow \sum_k \left( \ddot{\gamma}^k + \sum_{i,j} \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j \right) \frac{\partial}{\partial x^k} = 0$$

$$\Rightarrow \ddot{\gamma}^k + \sum_{i,j} \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

- 2nd order ODE's
- non-linear $(\dot{\gamma}^i \dot{\gamma}^j)$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_k \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$$
Example 0 \quad M = \mathbb{R}^n \quad g_0 = \sum_i dx^i \otimes dx^i

\begin{align*}
(q_{ij}) &= I_n = (\delta_{ij}) \\
\Gamma^k_{ij} &= 0 \quad \forall \ i,j,k.
\end{align*}

The Riemannian connection for $g_0$ is

$$
\nabla_{\dot{\gamma}} \gamma = \sum_i \frac{d\gamma^i}{dx^j} \dot{\gamma}^j.
$$

The geodesic equations for $g_0$ are:

$$
\ddot{\gamma}^k = 0
$$

$$
\begin{align*}
\gamma^k(t) &= \gamma^k(0) + \dot{\gamma}^k(0) t \\
\gamma(t) &= \gamma(0) + \dot{\gamma}(0) t \quad \text{a straight line.}
\end{align*}
$$
inherits a metric from standard metric on 1123