Surfaces of Revolution

Consider \( c: (0, b) \to \mathbb{R}^3 \)
\[ t \mapsto (F(t), 0, G(t)) \]

Assume that \( F(t) \geq 0 \) and \( G'(t) \geq 0 \)

Rotating \( c \) around the \( z \)-axis we get a surface
\[ S = \left\{ \left( F(x^2) \cos x^1, \ F(x^2) \sin x^1, \ G(x^2) \right) \mid x^1 \in [0, \pi), \ x^2 \in (0, b) \right\} \]
\[ S = S^1 \times (0, 1) \]

Ex. \( c(t) = (t, 0, +) + \varepsilon (0, 1) \)

\[ S \text{ is an open cone} \]
The Christoffel symbols of the Riemannian connection are

The other 4 are zero

The geodesic equations are

Need to solve using tricks

Assume Ito Equations

$S$ inherits a metric $g$ from the standard metric $g_0$ on $\mathbb{R}^3$

$T_1S = \text{Span} \{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \} \subset \text{Span} \{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \}$

$$\frac{\partial}{\partial x^1} = \frac{\partial x}{\partial x^1} \frac{\partial}{\partial x} + \frac{\partial y}{\partial x^1} \frac{\partial}{\partial y} + \frac{\partial z}{\partial x^1} \frac{\partial}{\partial z}$$

$$= -F(x^1) \sin x^1 \frac{\partial}{\partial x} + F(x^1) \cos x^1 \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial x^2} = F(x^2) \cos x^1 \frac{\partial}{\partial x} + F(x^2) \sin x^1 \frac{\partial}{\partial y} + G(x^2) \frac{\partial}{\partial z}.$$

$$g(x^i, x^j) = \sum g_{ij}(x^i, x^j) \, dx^i \otimes dx^j$$

$$g_{ij}(x^i, x^j) = g_0(x^i, x^j) \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$$

$S = S^2 \setminus \text{poles}$
\[ g_{11} (x^1, x^2) = (F(x^2))^2 \]
\[ g_{22} (x', x^2) = (F'(x^2))^2 + (G'(x^2))^2 \]
\[ g_{12} = g_{21} = 0 \]

With this, we can compute the Christoffel symbols of \( \nabla \).

\[ \Gamma^1_{12} = \Gamma^1_{21} = \frac{F'(x^2)}{F(x^2)} \quad \Gamma^1_{11} = \Gamma^1_{22} = 0 \]

\[ \Gamma^2_{11} = -\frac{FF'}{(F')^2 + (G')^2}, \quad \Gamma^2_{22} = \frac{FF'' + 6G'G''}{(F')^2 + (G')^2}, \quad \Gamma^2_{12} = \Gamma^2_{21} = 0. \]

\[ \ddot{y}^1 + 2 \frac{F'(y^2)}{F(y^2)} \dot{y}^1 \dot{y}^2 = 0 \]

\[ \ddot{y}^2 = -\frac{F'}{(F')^2 + (G')^2} (\dot{y}^1)^2 + \frac{FF'' + G'G''}{(F')^2 + (G')^2} (\dot{y}^2)^2 = 0 \]

The corresponding geodesic on \( S \) is

\[ (F(y^2) \cos \dot{y}^1, F(y^2) \sin \dot{y}^1, G(y^2)) \]
Theorem 1 \((g, Y(t))\) is geodesic iff \(\dot{Y} \parallel g\) is constant.

\[
\begin{align*}
\dot{Y} &\parallel g \\
1 &\Leftrightarrow \ddot{y}^2 + \frac{F'' + F'C''}{F'^2 + C'^2} = 0
\end{align*}
\]

which is (2)
Thm 2 \( (\dot{\mathbf{r}}(t), k) \) is a geodesic iff \( \|\dot{\mathbf{r}}\|_g = \text{const} \) and \( f'(k) = 0 \).

\[ \|\dot{\mathbf{r}}\|^2 = \text{const} \Rightarrow \dot{f}(k) \langle \dot{\mathbf{r}}^1, \dot{\mathbf{r}}^1 \rangle = \text{const} \]

\[ \Rightarrow \dot{\mathbf{r}}^1 = \pm \frac{\text{const}}{f'(k)} \]

\[ \Rightarrow \mathbf{r}^1(t) = \pm \frac{\text{const}}{f'(k)} t + \mathbf{r}^1(0) \]

Check

\( \ddot{\mathbf{r}}^1 + 2 \dot{\mathbf{r}}^1 \dot{\mathbf{r}}^2 \frac{f'(\mathbf{r}^2)}{f(\mathbf{r}^2)} = 0 \)

\[ \uparrow \quad \uparrow \]

\( \downarrow \quad \downarrow \)

\[ \dot{\mathbf{r}}^2 = (\dot{\mathbf{r}}^1)^2 \frac{ff'}{(f')^2 + (g')^2} + (\dot{\mathbf{r}}^2)^2 \frac{ff' + gg''}{(f')^2 + (g')^2} \]

\[ \downarrow \quad \uparrow \]

\[ \downarrow \quad \downarrow \]

\[ = 0 \quad \text{iff} \quad f'(k) = 0. \]

Ex

\[ \text{geo\-desic} \]
Ex. Consider the following geodesic triangle on $S^2$

- The sum of interior angles is $3(\frac{\pi}{2})$.
- Compare this to $\pi$ for triangles in $(\mathbb{R}^2, g_o)$ and $< \pi$ for triangles in $(\mathbb{H}, g)$ (See Homework 10, 2b)
- The theorem which explains this phenomenon is the Gauss-Bonnet Thm. It involves Curvature