1. Let \( X \) be a fixed connected topological surface. Show that a Riemann surface structure on \( X \) is determined by the data of a single non-constant meromorphic function on \( X \). More precisely, if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are maximal Riemann surface atlases on \( X \), and there is a function \( f \) on \( X \) that is meromorphic with respect to both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), then \( \mathcal{A}_1 = \mathcal{A}_2 \).

**Solution:** A nonconstant meromorphic function \( f \) on \( X \) extends to a map \( F : X \rightarrow \mathbb{C}_\infty \). The hypothesis amounts to saying that \( F \) is holomorphic with respect to both \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \).

Pick \( p \in X \), let \( q = F(p) \in \mathbb{C}_\infty \). Choose a local coordinate \( z \) on \( \mathbb{C}_\infty \) centered at \( q \). Since \( F \) is holomorphic with respect to \( \mathcal{A}_1 \), the local normal form lemma for holomorphic maps says that there is a local coordinate \( w_1 \) from the atlas \( \mathcal{A}_1 \) centered at \( p \) such that \( F \) takes the form \( z = w_1^m \), where \( m = \text{mult}_p(F) \). Similarly, since \( F \) is holomorphic with respect to \( \mathcal{A}_2 \), there is a local coordinate \( w_2 \) from the atlas \( \mathcal{A}_2 \) such that \( F \) takes the form \( z = w_2^m \).

Thus we have \( w_1^m = w_2^m \), and this equation is valid in an open set containing \( p \). Both \( w_1 \) and \( w_2 \) are continuous with respect to the topology of \( X \), so we conclude that there is an \( m \)-th root of unity \( \zeta \) such that \( w_1 = \zeta w_2 \) is valid in an open set containing \( p \). Thus \( w_1 \) is a holomorphic function of \( w_2 \) and vice versa. Since \( p \in X \) was arbitrary, we find that for each \( p \in X \), there are local coordinates (charts) from \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) that are compatible with each other. It follows that any coordinate from \( \mathcal{A}_1 \) is compatible with any coordinate from \( \mathcal{A}_2 \), so \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are equivalent atlases, and hence equal by maximality.

2. Consider \( \text{Aut}(\mathbb{C}_\infty) \), the group of holomorphic automorphisms of the Riemann sphere \( \mathbb{C}_\infty \). Prove using the results from this course that \( \text{Aut}(\mathbb{C}_\infty) \) consists precisely of the Möbius transformations

\[
\mu_{a,b,c,d}(z) = \frac{az + b}{cz + d}
\]

where \( a, b, c, d \in \mathbb{C} \) satisfy \( ad - bc \neq 0 \). (In this formula, it is to be understood that if the denominator vanishes, the value is \( \infty \), and if \( z = \infty \), the value is \( a/c \).)

**Solution:** The group \( \text{Aut}(\mathbb{C}_\infty) \) consists of isomorphism \( \varphi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \), and an isomorphism is the same as a degree 1 holomorphic map. Also, a holomorphic map to \( \mathbb{C}_\infty \) is the same as a meromorphic function, and meromorphic functions on \( \mathbb{C}_\infty \) are rational functions. Thus \( \varphi(z) = p(z)/q(z) \) where \( p(z) \) and \( q(z) \) are polynomials with no common factor. Since \( \varphi \) is meant to have degree 1, it must have exactly one simply zero and one simple pole (either of which may be located at \( \infty \) in the domain). This implies that \( p(z) \) and \( q(z) \) have degree at most 1. It follows that \( \varphi(z) \) is a ratio of two linear functions (either of which may be constant), so \( \varphi(z) = \mu_{a,b,c,d}(z) \) for some numbers \( a, b, c, d \). The condition \( ad - bc \neq 0 \) means that the numerator and denominator are not proportional, and is necessary in order for \( \mu_{a,b,c,d} \) to be a nonconstant map.

Conversely, each rational function \( \mu_{a,b,c,d}(z) \) with \( ad - bc \neq 0 \) does define a nonconstant holomorphic map \( \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \). If \( a \neq 0 \), This map has a single zero at \( z = -b/a \), and has order 1 there. Thus \( \deg \mu_{a,b,c,d} = 1 \). If \( a = 0 \), then \( c \neq 0 \), so \( \infty \) maps to 0, and \( \mu_{a,b,c,d} \) has a single pole at
z = -d/c, which is also simple, so again degμ_{a,b,c,d} = 1. Thus in all cases μ_{a,b,c,d} is an isomorphism.

3. Consider the Riemann surface X defined for (x, y) ∈ C^2 by the equation y^2 = x(x-1)(x-2). If R(x, y) = P(x, y)/Q(x, y) is a rational function in two variables whose denominator does not vanish identically on X, then ω = R(x, y)dx is a meromorphic 1-form on X. Show that if R(x, y) = p(x)/y, with p(x) a polynomial, then ω is holomorphic on X.

Solution: It suffices to show that dx/y is holomorphic on X. Since x defines a holomorphic function on X, p(x) is a holomorphic function on X. If dx/y is holomorphic, so is the product ω = p(x)dx/y.

So we claim dx/y is holomorphic on X. First of all dx is holomorphic on X since it is d of a holomorphic function. On the open subset U of X where y ≠ 0, the function 1/y is holomorphic, so dx/y is holomorphic on U. It remains to consider the points where y = 0. Differentiating the equation for X, we find 2ydy = h’(x)dx, where h(x) = x(x-1)(x-2). When y = 0, h(x) = 0, and so h’(x) ≠ 0 since h has no repeated roots. Thus we may solve for dx as dx = (2y/h’(x))dy. By the implicit function theorem, y is a local coordinate on X near any point where y = 0. Near such a point, dx/y = (2/h’(x))dy is the local expression of dx/y in the coordinate, and the function (2/h’(x)) is holomorphic near this point. This completes the proof that dx/y is holomorphic on all of X.

4. Let K be the field of meromorphic functions on C that are ℤ-periodic, meaning that f(z + 1) = f(z) for all z not a pole of f. Show that K is isomorphic to the field of meromorphic functions on C \ {0}. Hint: e^{2πi z}.

Solution: Consider the holomorphic map F : C → C \ {0} given by F(z) = e^{2πi z}. This is a surjective homomorphism of groups (addition on C, multiplication on C \ {0}), with kernel ℤ, and so it induces a bijection $\overline{F} : C/\mathbb{Z} → C \ {0}$.

Denote by M the field of meromorphic functions on C \ {0}. We define a homomorphism $φ : M → K$ by precomposition with F: $φ(f) = f ∘ F$, or $φ(f)(z) = f(e^{2πi z})$ for $f ∈ M$. Clearly $φ(f)(z + 1) = φ(f)(z)$, so $φ(f) ∈ K$. Since $φ$ is a homomorphism of fields, it is necessarily injective.

It remains to show that $φ$ is surjective. For this, let $g ∈ K$ be an arbitrary ℤ-periodic meromorphic function. We can regard it as a map $g : C → C_∞$. Because g is periodic, it is constant on cosets of ℤ in C, and so descends to a function $\overline{g} : C/\mathbb{Z} → \mathbb{C}_∞$. Using the bijection $\overline{F}$, we obtain a function $f = \overline{g} ∘ (\overline{F})^{-1} : C \ {0} → C_∞$. Clearly $φ(f) = g$, so it remains to show that $f ∈ M$.

Let $p ∈ C \ {0}$. In a small neighborhood of p, we may choose a single-valued and holomorphic branch $ln(w)$ of the logarithm function. Then $(2πi)^{-1}ln(w)$ is a local inverse to F, and so $f(w) = g((2πi)^{-1}ln(w))$. Since ln(w) is holomorphic and g is meromorphic, f is meromorphic near p. Since p was arbitrary, we are done.

5. Consider the one-parameter family of projective plane curves, depending on $ψ ∈ C$,

$$X_ψ = \{[x : y : z] ∈ \mathbb{CP}^2 \mid F_ψ(x, y, z) = 0\},$$

where $F_ψ(x, y, z) = x^3 + y^3 + z^3 - 3ψ xyz$. Find all parameter values $ψ$ for which $X_ψ$ is singular, and find the singular points of those curves.

Solution: Taking derivatives, we find that the equations for a singular point are $3x^2 - 3ψyz = 0$, $3y^2 - 3ψxz = 0$, and $3z^2 - 3ψxy = 0$. (These equations also imply vanishing of $F_ψ$ by Euler’s
lemma.) Multiply first equation by $x$, second by $y$, and third by $z$ to find $x^3 = y^3 = z^3 = \psi xyz$. Since we are working in projective space, $x, y, z$ cannot all be zero, in fact none of them can be zero. Thus we are free to normalize $z$ to be 1, and so the system becomes $x^3 = y^3 = \psi xy = 1$.

This system has nine total solutions: we let $x$ and $y$ be arbitrary cube roots of unity, and then let $\psi$ be the reciprocal of their product. In each case, $\psi$ is a cube root of unity. Let $\zeta = e^{2\pi i / 3}$, so the cube roots of unity are $1, \zeta, \zeta^2$.

