Examples of Riemann surfaces.

To construct examples in a concise manner, it is useful to streamline the process so that we only have to check a minimum of things.

**Point-set topology:** A space $X$ is **regular** if any point $x_0 \in X$ and closed set $Y \subseteq X$ with $x_0 \not\in Y$ can be separated by open sets $(\exists U, V \text{ open}) x_0 \in U$, $Y \subseteq V$, $U \cap V = \emptyset$)

**Lemma** Let $X$ be a Hausdorff space, and suppose $A = \{ (U_x, \varphi) \}^x$ is a countable atlas (of charts $\varphi : U_x \to V_x \subseteq \mathbb{C}$).

Then necessarily $X$ is metrizable and separable.

**Proof:** The existence of the charts $\varphi : U_x \to V_x$ shows that $X$ is locally metrizable, since each $V_x \subseteq \mathbb{C}$ is metrizable. This implies that $X$ is regular.

Also, each $U_x$ is 2nd countable. Since there only countably many $U_x$'s, $X$ is 2nd countable. Apply Urysohn metrization theorem: Hausdorff + regular + 2nd countable $\Rightarrow$ metrizable.

Lastly, for a metrizable space the conditions of being separable and being 2nd countable are equivalent.

**Remark:** Every (separable metrizable) Riemann surface admits a countable atlas; take charts containing each point of a countable dense subset.

So to define a Riemann surface, it suffices to construct a Hausdorff space and a countable atlas on it.
Next, we note that since \( c_\alpha : U_\alpha \to V_\alpha \subset C \) is a homeomorphism onto an open set in \( C \), the charts can be used to determine the topology.

Here is the prescription:

1. Take a set \( X \).
2. Take a countable collection of sets \( U_\alpha \subset X \) that cover \( X \) \( \bigcup_\alpha U_\alpha = X \).
3. For each \( \alpha \), take an open set \( V_\alpha \subset C \) and a bijection \( c_\alpha : U_\alpha \to V_\alpha \).
4. Define a topology \( T_\alpha \) on \( U_\alpha \) by declaring that \( c_\alpha \) is a homeomorphism.
5. Define a topology \( T_X \) on \( X \) by declaring \( U \subset X \) is open iff \( U \cap U_\alpha \in T_\alpha \) for every \( \alpha \).
6. It is not necessarily true that \( T_X \) is the same as the subspace topology on \( U_\alpha \) as a subspace of \( (X, T_X) \). However, this is true under the following condition:
   \( \forall \alpha, \beta \) \( c_\beta (U_\alpha \cap U_\beta) \) is open in \( C \).
   We must check this condition. Then \( (U_\alpha, c_\alpha) \) is a chart.
7. Check that the charts \( \{(U_\alpha, c_\alpha)\}_{\alpha} \) are pairwise compatible.
8. Check that \( X \) is Hausdorff.

In short, we need to supply set theoretic data \( X, U_\alpha, c_\alpha \), and then check the things mentioned in 6,7,8.
The projective line. \( \mathbb{C}^2 = \text{2-dimensional complex vector space} \)

Define set \( X = \mathbb{C}P^1 = \{ L \subset \mathbb{C}^2 \mid L \text{ is a 1-dimensional C-subspace} \} \)

if \( (z,w) \neq (0,0) \), then \( (z,w) \) is a 1-d subspace.

We denote this subspace by \( [z:w] \).

Observe \( [z:w] = [\lambda z: \lambda w] \) for any \( \lambda \in \mathbb{C} \setminus \{0\} \).

And indeed \( [z:w] = [z':w'] \) iff \( [z:w] = [\lambda z: \lambda w] \) for some \( \lambda \).

Take subsets \( U_0 = \{ [z:w] \mid z \neq 0 \} \subset \mathbb{C}P^1 \)
\( U_1 = \{ [z:w] \mid w \neq 0 \} \subset \mathbb{C}P^1 \)

Define \( \varphi_0 : U_0 \to \mathbb{C} \quad \varphi_0([z:w]) = w/z \)
\( \varphi_1 : U_1 \to \mathbb{C} \quad \varphi_1([z:w]) = z/w \)

These are well-defined since rescaling \( z \) and \( w \) by same \( \lambda \) doesn't change ratio. They are also bijections:
\( \varphi_0^{-1}(x) = [1:x] \quad \varphi_1^{-1}(x) = [x:1] \)

Need to check: \( \varphi_0(U_0 \cap U_1) = \mathbb{C} \setminus \{0\} \)
\( \varphi_1(U_0 \cap U_1) = \mathbb{C} \setminus \{0\} \) which are open in \( \mathbb{C} \). \( \checkmark \)

Transition function \( \varphi_1 \circ \varphi_0^{-1}(x) = \varphi_1([1:x]) = \frac{1}{x} \) which is holomorphic as a map \( \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\} \) \( \checkmark \)

Is \( \mathbb{C}P^1 \) Hausdorff? Let \( p,q \in \mathbb{C}P^1 \). If both in same chart, we can separate them by disks in that chart. Only other possibility is \( p = [1:0] \) and \( q = [0:1] \). Then small disk around \( p \) in \( U_0 \) and around \( q \) in \( U_1 \) do the trick. \( \checkmark \)
Graph of a holomorphic function. Let $V \subset \mathbb{C}$ be open, and let $g : V \rightarrow \mathbb{C}$ be a holomorphic function. Let $X = \{ (z, g(z)) \mid z \in V \} \subset \mathbb{C}^2$.

Define a chart $U = X \times \mathbb{C}$, $\varphi : U \rightarrow V$, $\varphi(z, w) = z$.

This is a bijection, as the inverse is $\varphi^{-1}(z) = (z, g(z))$.

Observe that $\varphi$ is a homeomorphism from the subspace topology on $X \subset \mathbb{C}^2$ to $V$. Everything we need to check is clear.

Riemann surface of a multivalued function:

Suppose $g$ is a "multivalued" holomorphic function. e.g. $g(z) = \log z$ or $g(z) = z^\alpha$. We can define a subset of $\mathbb{C}^2$,

$X = \{ (z, w) \mid w \text{ is a value of some single-valued branch of } g(z) \} \subset \mathbb{C}^2$.

We can define charts on $X$ by restricting to $z \in V$ where $g$ has a single-valued branch $\tilde{g}$ on $V$. Set $U = \{ (z, g(z)) \mid z \in V \}$, and $\varphi : U \rightarrow V$ projects to first coordinate.

The topology this process defines on $X$ does not necessarily agree with the subspace topology of $X \subset \mathbb{C}^2$.

If we apply this to $g(z) = \log z$, then we get

$X = \{ (z, w) \mid z = \exp(w) \} \subset \mathbb{C}^2$, which is a graph "the other way."

If we apply this to $g(z) = \sqrt[3]{z}$ we get

$X = \{ (z, w) \mid z = w^3, z \neq 0 \}$, which is missing because there is no single valued branch of $\sqrt[3]{z}$ around $z = 0$. $z = 0$ is called a branch point.

Historically, this was a motivation for the theory of Riemann surfaces.
Affine plane curves: Let \( f(z, w) \in \mathbb{C}[z, w] \) be a polynomial in two variables with complex coefficients.

The zero locus of \( f \) is the set \( X = \{ (z, w) \mid f(z, w) = 0 \} \subset \mathbb{C}^2 \).

**Implicit function theorem:** Suppose \((z_0, w_0) \in X\), and

\[
\frac{\partial f}{\partial w}(z_0, w_0) \neq 0.
\]

Then there is a unique holomorphic function \( g(z) \) defined in an open set \( V \ni z_0 \) such that \( g(z_0) = w_0 \) and \( f(z, g(z)) = 0 \) for all \( z \in V \).

Near \((z_0, w_0)\), \( X \) coincides with the graph of \( g(z) \).

**How to remember the theorem:**

Differentiate \( f(z, w) = 0 \)

\[
\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial w} dw = 0
\]

Try to solve for \( dw \) in terms of \( dz \):

\[
dw = -\left(\frac{\partial f}{\partial w}\right)^{-1} \frac{\partial f}{\partial z} dz
\]

Need \( \frac{\partial f}{\partial w} \neq 0 \) to do this.

IFT says that, if you can solve for \( dw \) in terms of \( dz \),

you can solve for \( w \) in terms of \( z \) **locally**.

Similarly:\n
\( \frac{\partial f}{\partial z} \neq 0 \) \( \Rightarrow \) solve for \( dz \) in terms of \( dw \)

\( \Rightarrow \) solve for \( z \) in terms of \( w \) **locally**.

The polynomial \( f(z, w) \) is **nonsingular** if, for every \((z_0, w_0) \in X = \{ f = 0 \}\),

at least one of \( \frac{\partial f}{\partial w}(z_0, w_0) \) and \( \frac{\partial f}{\partial z}(z_0, w_0) \) is non-zero.

I.e., \( f, \frac{\partial f}{\partial w}, \frac{\partial f}{\partial z} \) do not all vanish at same point.
Proposition let $f(z,w)$ be a nonsingular polynomial. Then $X = \{ (z,w) \mid f(z,w) = 0 \} \subset \mathbb{C}^2$ has a Riemann surface structure.

The atlas is constructed by systematically applying the implicit function theorem. At points where $\frac{\partial f}{\partial w} \neq 0$, we find $V \subset \mathbb{C}$ and $g : V \to \mathbb{C}$ such that locally

$X$ is the graph $\{(z, g(z)) \mid z \in V \}$ (projection to first coordinate).

At points where $\frac{\partial f}{\partial z} \neq 0$, we find $V \subset \mathbb{C}$ and $h : W \to \mathbb{C}$ so that locally

$X$ is the "other way" graph $\{(h(w), w) \mid w \in W \}$ (projection to second coordinate).

It is easy to see these charts are compatible, see Miranda.

Picture

\[ f(z,w) = z^2 + w^2 - 1 \]

\[ \frac{\partial f}{\partial z} = 2z \quad \frac{\partial f}{\partial w} = 2w \]

so nonsingular.

"$W = \pm \sqrt{1-z^2}$" let $V = \mathbb{C}^2$

\[ \begin{array}{ccc}
\pm \sqrt{1-z^2} & \cdots & 0 \\
0 & \cdots & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{array} \]

branch cuts

Thus $\pm \sqrt{1-z^2}$ has two single valued branches in this domain.

By moving the branch cuts, can cover all points except $w = 0, z = \pm 1$.

Swap roles of $w, z$, can use similar charts to cover all points except $z = 0, w = \pm 1$. Ultimately we cover everything.