Holomorphic maps and topology of surfaces

Today, all Riemann surfaces are assumed connected.

Recall local normal form of a holomorphic map:
let $F:X \to Y$ be holomorphic map which is not constant.
Pick $p \in X$. There is a unique integer $m \geq 1$
such that there are local coordinates $z$ centered at $p \in X$
and $w$ centered at $F(p) \in X$ such that $F$ takes the form
$$z \mapsto w = z^m$$

We saw existence last time. The number $m$ is unique
because it can also be characterized as the number of
preimages of a point near $F(p)$ that are near $p$.

\[
\begin{array}{c}
\circ \quad \cdots \quad \cdots \quad \cdots \\
| \quad p \quad | \quad \cdots \\
\end{array}
\quad \rightarrow 
\begin{array}{c}
\circ \quad \cdots \quad \cdots \quad \cdots \\
| \quad F(p) \quad | \quad \cdots \\
\end{array}
\]

\[
|F^{-1}(w) \cap \text{nbhd}(p)| = m.
\]

Definition: With notation as above, $m$ is called the
multiplicity of $F$ at $p$. \[m = \text{mult}_p(F)\]

Note $m \geq 1$ always. To compute $\text{mult}_p(F)$, we can actually
use any coordinate system.

Lemma: Let $z$ be a local coordinate near $p$, $p \leftrightarrow z_0$
let $w$ be a local coordinate near $F(p)$, $F(p) \leftrightarrow w_0$.
$F$ has coordinate representation $w = h(z)$. 
Lemma cont’d: Then $\text{mult}_p(F) = 1 + \text{ord}_{z_0} \left( \frac{dh}{dz} \right)$.

Alternatively, $\text{mult}_p(F)$ equals the lowest positive exponent in the power series for $h$.

If $h(z) = h(z_0) + \sum_{i=m}^{\infty} c_i (z-z_0)^i$ and $c_m \neq 0$ then $\text{mult}_p(F) = m$.

Proof: the "centered" coordinates are $\tilde{z} = z - z_0$ and $\tilde{w} = w - w_0$, and the map is

$$\tilde{w} = \sum_{i=m}^{\infty} c_i \tilde{z}^i = \tilde{z}^m \left( \sum_{j=0}^{\infty} \frac{c_{i+m}}{g(\tilde{z})} \right) \quad g(0) \neq 0$$

$$g(z) \text{ holomorphic} \quad \text{mult}_p(F) = m.$$

Corollary: $\exists p \in X \mid \text{mult}_p(F) \geq 2$ if $X$ is discrete.

If $X$ is compact, this set is finite.

Proof: locally this set is given as the vanishing locus of a holomorphic function $(\frac{dh}{dz}$ in above).

Corollary follows from discreteness of zeros.

Definition: $p \in X$ such that $\text{mult}_p(F) \geq 2$ is called a ramification point. The image $F(p)$ is called a branch point.

Lemma: Let $f$ be a nonconstant meromorphic function on $X$, and let $F: X \rightarrow \mathbb{C} \cup \{\infty\}$ be corresponding holomorphic map to the Riemann sphere.
Lemma cont'd

- If \( p \in X \) is a zero of \( f \), then \( \text{mult}_p(F) = \text{ord}_p(f) \).
- If \( p \in X \) is a pole of \( f \), then \( \text{mult}_p(F) = -\text{ord}_p(f) \).
- Otherwise \( \text{mult}_p(F) = \text{ord}_p(f - f(p)) \).

Degrees

Let \( X \) be compact, \( F: X \to Y \) holomorphic and nonconstant. So \( Y \) is compact and \( F \) is surjective.

For \( y \in Y \), define local degree:

\[
d_y(F) = \sum_{p \in F^{-1}(y)} \text{mult}_p(F).
\]

Proposition

\( d_y(F) \) is independent of \( y \). We call it the degree of \( F \).

\[
\deg(F) = d_y(F) \quad \text{for any/all } y.
\]

Proof: Since \( Y \) is connected, it suffices to show that \( d_y(F) \) is locally constant: \( (y \in Y) \exists \text{ U open } (d_y(F) \text{ is constant on U}) \).

Choose \( U \subseteq Y \) disk around \( y \) so small that \( F^{-1}(U) \subseteq X \) has separate connected components for each \( p \in F^{-1}(y) \).

As we move from \( y \) to \( y' \) near \( y \), we see that each preimage \( p_i \in F^{-1}(y) \) splits into \( \text{mult}_p(F) \) separate preimages, each with multiplicity 1.

(Follows from normal form near \( p_i \))

Thus \( d_y(F) = \sum_{p \in F^{-1}(y)} \text{mult}_p(F) = \sum_{p \in F^{-1}(y')} 1 = d_{y'}(F) \),

for all \( y' \in U \). □
Corollary: A holomorphic map $F: X \to Y$ is an isomorphism iff $\deg(F) = 1$.

Proof: $\deg(F) = 1 \iff F$ is bijective.

Knew $F$ is surjective since $X$ compact, so $F$ is bijective.
Bijective $\Rightarrow$ holomorphic $\Rightarrow$ isomorphism.

Corollary: Suppose $X$ is compact and $f$ is a meromorphic function with a single simple pole. Then $X \cong \mathbb{C}^g$.

Proof: Let $F: X \to \mathbb{C}^g$ be holomorphic map corresponding to $f$.

Then $d_0(F) = 1$, so $\deg(F) = 1$, so $F$ is isomorphic.

Proposition: Let $f$ be a nonconstant meromorphic function on a compact Riemann surface $X$.

Then $\sum_{p \in X} \text{ord}_p(f) = 0$.

Proof: Let $F: X \to \mathbb{C}^g$ be corresponding map.

Then $F^{-1}(0) = \text{zeros of } f$

$F^{-1}(\infty) = \text{poles of } f$

$d_0(F) = \sum_{p \in F^{-1}(0)} \text{mult}_p(F) = \sum_{p \in \text{zeros}(f)} \text{ord}_p(f)$

$d_\infty(F) = \sum_{p \in F^{-1}(\infty)} \text{mult}_p(F) = \sum_{p \in \text{poles}(f)} -\text{ord}_p(f)$

Then $\sum_{p} \text{ord}_p(f) = \sum_{p \in \text{zeros}(f)} \text{ord}(f) + \sum_{p \in \text{poles}(f)} \text{ord}_p(f)$

$= d_0(F) - d_\infty(F) = \deg(F) - \deg(f) = 0$.
Euler characteristic of surfaces. Now I will assert some facts
from the theory of topological surfaces.

1. Each compact orientable surface is homeomorphic
to a "torus with $g$ holes" for some $g \geq 0$.
   $g$ is called the genus.
   
   \[ g = 0 \quad g = 1 \quad g = 2 \quad g = 3 \quad \ldots \]

2. To a compact surface $S$ we may associate a number $X(S)$
called the Euler characteristic. This number is invariant
with respect to homeomorphism and it can be computed
as follows: Triangulate $S$ (divide $S$ into triangles)

\[
\begin{align*}
\text{Then } X(S) &= V - E + F \\
V &= \text{vertices} \\
E &= \text{edges} \\
F &= \text{faces}.
\end{align*}
\]

[All triangulations give same $X$.]

3. If $S_g$ is a surface of genus $g$, then $X(S_g) = 2 - 2g$

\[
\begin{align*}
V &= 2 \\
E &= 4g + 2g \\
F &= 4g
\end{align*}
\]
\[
X = 2 - 6g + 4g = 2 - 2g.
\]
Hornitz formula: Let \( F : X \to Y \) be nonconstant, \( X, Y \) compact.

\[
\chi(X) = \deg(F) \chi(Y) - \sum_{p \in X} [\text{mult}_p(F) - 1].
\]

**Proof.** Take a triangulation of \( Y \) so fine that each branch point of \( F \) is a vertex of \( Y \). The preimage of a triangle in \( Y \) is then several triangles in \( X \).

Now \((\text{Faces in } X) = \deg(F)(\text{Faces in } Y)\)

\((\text{Edges in } X) = \deg(F)(\text{Edges in } Y)\)

But \((\text{Vertices in } X) \leq \deg(F)(\text{Vertices in } Y)\),

since \( \deg(F)(\text{Vertices in } Y) \) overcounts the preimages of the branch points.

If \( q \) is a vertex in \( Y \), then

\[
|F^{-1}(q)| = \sum_{p \in F^{-1}(q)} 1 = \deg(F) + \sum_{p \in F^{-1}(q)} [1 - \text{mult}_p(F)]
\]

Thus \((\text{Vertices in } X) = \deg(F)(\text{Vertices in } Y) - \sum_{p \in X} [\text{mult}_p(F) - 1]\)

This formula relates the genus of \( X \) and \( Y \):

\[
2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\text{mult}_p(F) - 1]
\]

**Corollary:** \( g(X) \geq g(Y) \) if \( \exists F : X \to Y \) nonconstant.