Finiteness theorem for \( M(X) \) and weak Riemann Roch.

Let \( X \) be a compact Riemann surface. \( X \) is algebraic so there is a nonconstant meromorphic function \( f \in M(X) \). The subfield generated by \( C \) and \( f \) is \( C(f) \subseteq M(X) \). Now \( f \) is transcendental over \( C \) (since \( C \) is algebraically closed) so \( C(f) \) is isomorphic to \( C(\infty) \), the field of rational functions.

We can also regard \( f \) as defining a map \( F : X \to \mathbb{C} \). Let \( d = \deg F = \deg (\operatorname{div} \alpha(f)) \) be the degree of this map. The subfield \( C(f) \subseteq M(X) \) is equal to \( \{ F^k g \mid g \in M(\mathbb{C}\infty) = C(\infty) \} \).

**Theorem** \( M(X) \) is a finite extension of \( C(f) \), whose degree is \[
\left[ M(X) : C(f) \right] = \deg (\operatorname{div} \alpha(f))
\]

The proof is broken into several propositions. The starting point is the estimate: If \( D \geq 0 \), then \( \ell(D) \leq \deg D + 1 \). \( \ell(D) = \dim L(D) \).

**Proposition** \( M(X) \) is algebraic over \( C(f) \).

**Proof:** If not, then there is some \( g \in M(X) \) which is transcendental over \( C(f) \). Thus \( f \) and \( g \) are algebraically independent over \( C \). There is some nonnegative divisor \( D \) such that \( f, g \in L(D) \) \( \left( \text{e.g. } D = \max(\operatorname{div} f, \operatorname{div} g) \right) \). Then \( f^i g^j \in L(nD) \) if \( i+j \leq n \).
Since $f, g$ algebraically independent, $\{ f^i g^j \}_{i,j}$ are linearly independent. So $L(nD) \geq \# \{ f^i g^j \mid i \geq 0, j \geq 0, i + j \leq n \} = \frac{(n+1)(n+2)}{2}$.

On the other hand, $L(nD) \leq \deg (nD) + 1 = n \deg D + 1$.

So $\frac{(n+1)(n+2)}{2} \leq n \deg D + 1$ for $n > 0$.

This is absurd since the left-hand side grows quadratically, and the right-hand side grows linearly.

**Lemma**: Let $A \in \text{Div}(X)$, and let $D = \text{div}_n(f)$ for $f \in M(X) \setminus C$. Then there exists $m > 0$ and $g \in M(X)$ such that $A - \text{div}(g) \leq mD$.

Moreover, $g$ can be taken to be $r(f)$ where $r \in \mathbb{C}[t]$.

**Proof**: Let $p_1, \ldots, p_k$ be the points with $A(p_i) \geq 1$ and that are not poles of $f$. Then $f(p_i) \in \mathbb{C}$ and we may form

$$g = \prod_{i=1}^{k} \left( f - f(p_i) \right)^{A(p_i)}$$

Then $A - \text{div}(g)$ is positive only at the poles of $f$.

So $A - \text{div}(g) \leq mD$ for some $m > 0$.

**Corollary**: Let $f, h \in M(X) \setminus C$. Then there exists $r \in \mathbb{C}[t]$ such that $r(f)h$ has no poles outside the poles of $f$.

And there exists $m$ such that $r(f)h \in L(mD)$, where $D = \text{div}_n(f)$.

**Proof**: Apply lemma with $A = -\text{div}(h)$.

Then $A - \text{div}(r(f)) \geq mD$ which means $\text{div}(h) + \text{div}(r(f)) \geq -nD$. \(\square\)
**Lemma**  
Let \( f \in \mathcal{M}(X) \setminus \mathbb{C} \), \( D = \text{div}_{\text{loc}}(f) \). Suppose \([\mathcal{M}(X) : \mathcal{C}(f)] \geq k\). Then \( (\exists m_0) (\forall m \geq m_0) (\ell(mD) \geq (m-m_0+1)k)\)  

[i.e., Growth rate of \( \ell(mD) \) is a least linear of slope \( k \).]

**Proof**  
Take \( g_1, \ldots, g_k \in \mathcal{M}(X) \) linearly independent over \( \mathcal{C}(f) \). By corollary, take \( r_i \in \mathbb{C}[t] \) s.t. \( h_i := r_i(f)g_i \) has poles only at poles of \( f \). Then \( h_i \) are lin. indep. over \( \mathcal{C}(f) \) also. \( \exists m_0 \) s.t. \( \forall i \ h_i \in \ell(m_0D) \)

For \( m \geq m_0 \), \( f_i h_j \in \ell(mD) \) if \( i + m_0 \leq m \), since \( f \in \ell(D) \). These are lin. indep. over \( \mathcal{C} \), so \( \ell(mD) \geq \# \{ f_i h_j \mid 0 \leq i \leq m-m_0, 1 \leq j \leq k \} = (m-m_0+1)k \). \( \square \)

**Proposition**  
\([\mathcal{M}(X) : \mathcal{C}(f)] \leq \text{deg} \ D \)  
(Where \( D = \text{div}_{\text{loc}}(f) \))

**Proof:**  
If \([\mathcal{M}(X) : \mathcal{C}(f)] \geq \text{deg} \ D + 1 \), then by lemma \( \ell(mD) \geq (m-m_0+1)(\text{deg} \ D + 1) = (\text{deg} \ D + 1)m + \text{const} \). But also \( \ell(mD) \leq m \text{deg} \ D + 1 = (\text{deg} \ D)m + \text{const} \). For large \( m \), these inequalities contradict each other. \( \square \)

**Proposition**  
\([\mathcal{M}(X) : \mathcal{C}(f)] \geq \text{deg} \ D \)  
(\( D = \text{div}_{\text{loc}}(f) \))

(This proposition completes the proof of the theorem.)

**Proof:**  
Write \( D = \sum_{i=1}^{k} n_i p_i \), \( n_i \geq 1 \).

For \( i = 1, \ldots, k \), \( j = 1, \ldots, n_i \), let \( g_{i,j} \in \mathcal{M}(X) \) have pole of order \( j \) at \( p_i \), and no poles or zeros at other \( p_i \). (Laurent series approx.)
The number of these functions is \( \sum_{i=1}^{k} n_i = \deg D \), so it suffices to show \( \xi, g_{ij} \) are lin. indep. \( i \neq j \) over \( C(f) \).

Suppose \( \sum c_{ij}(f) g_{ij} = 0 \) is a \( C(f) \)-linear relation.

By clearing denominators, we may assume \( c_{ij}(f) \in C[t] \).

The only poles of \( c_{ij}(f) \) are at \( p_1, \ldots, p_k \), and
\[
\text{ord}_{p_k}(c_{ij}(f)) = n_k \cdot \deg c_{ij}.
\]

Look at the terms where \( \deg c_{ij} \) is maximal, and among those, choose one such that \( j \) is maximal.
This term is \( c_{i_0j_0}(f) g_{i_0j_0} \) for some \( (i_0, j_0) \). WLOG \( i_0 = 1 \).

Now divide through by \( c_{1j_0}(f) f ) \) to get \( \sum d_{ij}(f) g_{ij} = 0 \)
with \( d_{1j_0} = 1 \). All \( d_{ij}(f) \) have \( \text{ord}_{p_k}(d_{ij}(f)) \geq 0 \)
because \( \deg c_{ij} \) was maximal, and also \( n_k | \text{ord}_{p_k}(d_{ij}(f)) \).

Now take \( \text{ord}_{p_1} \) of the terms in the relation.
If \( i \neq 1 \), \( \text{ord}_{p_1}(d_{ij}(f) g_{ij}) \geq 0 \)
For \( i = 1 \), \( \text{ord}_{p_1}(d_{1j}(f) g_{1j}) = \text{ord}_{p_1}(d_{1j}(f)) + \text{ord}_{p_1}(g_{1j}) = \text{ord}_{p}(d_{1j}(f)) - j \)
Since \( \text{ord}_{p}(d_{1j}(f)) \) is a multiple of \( n_1 \) and \( 1 \leq j \leq n_1 \),
This term has negative order iff \( d_{1j}(f) \) is a constant.
There is such a term, namely \( d_{1j_0}(f) g_{1j_0} = g_{1j_0} \) with \( \text{ord} - j_0 \)
By construction, \( \mathcal{M} \) is maximal subject to this property, so there can be no other term that can cancel this pole of order \( -\mathcal{M} \) at \( p_1 \). This contradicts the linear relation that was assumed. 

Back to \( H^1(D) \) and Riemann-Roch

\[
\mathcal{M} = \mathcal{M}(x)
\]

\[
R = \left\{ r = (r_p)_{p \in X} \mid r_p \in \mathcal{M}, \quad \text{ord}_p(r) > 0 \text{ for all but finitely many } p \right\}
\]

\[
L(D) = \left\{ f \in \mathcal{M} \mid \forall p \quad \text{ord}_p(f) + D(p) \geq 0 \right\}
\]

\[
R(D) = \left\{ r \in R \mid \forall p \quad \text{ord}_p(r_p) + D(p) \geq 0 \right\}
\]

\[
\alpha : \mathcal{M} \to R, \quad \alpha(f) = r \quad \text{ s.t. } r_p = f \text{ for all } p.
\]

\[
\alpha_D : \mathcal{M} \to R/R(D), \quad H^0(D) = \ker \alpha_D = L(D)
\]

\[
H^1(D) = \text{coker } \alpha_D = R/(R(D) + \alpha(M))
\]

We have exact sequence for any \( D \):

\[
0 \to \mathcal{M}/L(D) \to R/R(D) \to H^1(D) \to 0
\]

Suppose \( D_1 \leq D_2 \) then \( L(D_1) \leq L(D_2) \) \( R(D_1) \leq R(D_2) \)

So we have diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{M}/L(D_1) \to & R/R(D_1) \to & H^1(D_1) \to & 0 \\
& & \downarrow & & \downarrow & \\
0 & \to & \mathcal{M}/L(D_2) \to & R/R(D_2) \to & H^1(D_2) \to & 0
\end{array}
\]

where vertical maps are all surjective.
Hence we have a short exact sequence of the kernels

$$0 \rightarrow L(D_2)/L(D_1) \rightarrow R(D_2)/R(D_1) \rightarrow H^1(D_1/D_2) \rightarrow 0$$

where \( H^1(D_1/D_2) := \ker (H^1(D_1) \rightarrow H^1(D_2)) \)

\( \dim L(D_2)/L(D_1) = \ell(D_2) - \ell(D_1) \)

\( \dim R(D_2)/R(D_1) = \deg(D_2) - \deg(D_1) \), this is because all that matters are the coefficients of \( \ell_p \) in degrees between \( D_1(p) \) and \( D_2(p) \).

**Conclusion:** \( H^1(D_1/D_2) \) is finite dimensional and \( \deg(D_2) - \deg(D_1) = \ell(D_2) - \ell(D_1) + \dim H^1(D_1/D_2) \)

**Theorem:** For any divisor \( D \), \( H^1(D) \) is finite dimensional.

**Proof:** Next time.

**Corollary:** \( \dim H^1(D_1/D_2) = \dim H^1(D_1) - \dim H^1(D_2) \)

Write \( h^1(D) := \dim H^1(D) \).

**Corollary:** If \( D_1 \leq D_2 \), \( \deg(D_2) - \deg(D_1) = \ell(D_2) - \ell(D_1) + h^1(D_1) - h^1(D_2) \)

or \( \deg D_2 - \ell(D_2) + h^1(D_2) = \deg D_1 - \ell(D_1) + h^1(D_1) \)

Since any two divisors are comparable to a third, we find that \( \deg D - \ell(D) + h^1(D) \) is constant overall divisors \( D \) on \( X \).
In particular, it takes the same value for $D$ as it does for $0$. \[ \deg 0 - \ell(0) + h'(0) = h'(0) - 1 \]

**Proposition (Weak Riemann-Roch)** for any $D \in \text{Div}(X)$

\[ \ell(D) - h'(D) = \deg D + 1 - h'(0) \]

Where we have yet to prove $H^1(D)$ is finite-dimensional.