The Canonical map.

\( X \) - compact Riemann Surface

\( w \) a meromorphic 1-form

\( K = \text{div}(w) \) a canonical divisor.

Recall \( \deg K = 2g - 2 \) and \( \ell(K) = g \).

\( g = 0 \quad |K| = \emptyset \)

\( g = 1 \quad K \sim O, \quad |K| = \{ O \} \)

Prove if \( g \geq 1 \) then \( K \) is base point free.

**Proof.** Need to show \( \ell(K-p) = g-1 \).

By RR

\[
\ell(K-p) - \ell(p) = \deg(K-p) + 1 - g = 2g - 3 + 1 - g = g - 2
\]

On a curve of genus \( g \geq 1 \), \( L(p) \) consists of constant functions. (If \( f \in L(p) \) is nonconstant, it gives a deg 1 map \( X \rightarrow \mathbb{P}^1 \).

So \( \ell(p) = 1 \), and \( \ell(K-p) = g-1 \).

Thus, if \( g \geq 1 \), \( \phi_K \) is a map \( \phi_K : X \rightarrow \mathbb{P}^{g-1} \).

If \( g = 1 \), \( \phi_K : X \rightarrow \mathbb{P}^0 = \{ p \} \), which is not interesting.

So in what follows, we usually assume \( g \geq 2 \).

Another description of \( \phi_K \): Let \( \mathcal{O}^1(X) = \{ \text{holomorphic 1-forms} \} \)

Define map

\[
\psi : X \rightarrow \mathbb{P}^*(\mathcal{O}^1(X))
\]

\[
p \mapsto \{ \omega \in \mathcal{O}^1(X) \mid \omega(p) = 0 \}
\]

that is \( p \) maps to the hyperplane of 1-forms that vanish at \( p \).

In suitable coordinates, \( \psi = \phi_p \).
Games 2: $\phi_K: X \to \mathbb{P}^1$ has degree $\deg K = 2$

Thus, the canonical map is a degree 2 map to $\mathbb{P}^1$.

Recall from the discussion of branched coverings that a surface is hyperelliptic iff it admits a degree 2 map to $\mathbb{P}^1$.

**Prop** Every genus 2 R.S. $X$ is isomorphic to a hyperelliptic surface, more specifically a degree 2 branched covering of $\mathbb{P}^1$ branched at 6 points.

$g \geq 3$: $\phi_K: X \to \mathbb{P}^{g-1}$ When is it an embedding?

When is it not an embedding? if $l(K-p-q) \neq l(K) - 2$

for some $p, q \in X$. Since $l(K) - 2 \leq l(K-p-q) \leq l(K-p) = l(K) - 1$

this means $l(K-p-q) = l(K) - 1 = g - 1$

With Riemann-Roch, this implies

$L(K-p-q) - L(p+q) = 2g - 4 + 1 - q = g - 3$

$g - 1 - L(p+q) = g - 3$

$L(p+q) = 2$

If $L(p+q) = 2$, there is a nonconstant $f \in L(p+q)$

$f$ may only have simple poles at $p$ and $q$, it cannot have only a single simple pole since $g > 0$, so $f$ has simple poles at $p$ and $q$. (If $p = q$, this reasoning shows $f$ has a double pole at $p$.). Thus $f: X \to \mathbb{P}^1$ has degree 2.

Thus $X$ is isomorphic to a hyperelliptic surface.

**Conclusion:** If $\phi_K: X \to \mathbb{P}^{g-1}$ is not an embedding, $X$ is hyperelliptic.
Prop: If \( X \) is hyperelliptic, \( \phi_k \) is not an embedding.

**Proof:** Let \( \pi: X \to \mathbb{P}^1 \) be a degree 2 map. 
Think of \( \pi \in \mathcal{M}^k(X) \) as a nonconstant meromorphic function. 
Let \( \text{div}_\pi = p+q \) be the divisor of poles. 
Thus \( \pi \in L(p+q) \). Since constants are in \( L(p+q) \), we find \( \ell(p+q) = 2 \). 
Since \( \ell(p+q) \leq \ell(p)+1 \), and \( \ell(p) = 1 \) we find \( \ell(p+q) = 2 \).
Thus \( \ell(K-p-q) = 2g-4 + 1 - q + \ell(p+q) = g-1 \) 
so \( \phi_k \) is not an embedding.

Thus for surfaces of genus \( g \geq 3 \), we have a dichotomy:

Exactly one of the following holds:

(a) \( \phi_k: X \to \mathbb{P}^{g-1} \) is an embedding.

(b) \( X \) is hyperelliptic.

If (a), then \( \phi_k: X \to \mathbb{P}^{g-1} \) embeds \( X \) as a projective curve of degree \( 2g-2 \).

E.g., if \( g = 3 \) and \( X \) is not hyperelliptic, \( \phi_k: X \to \mathbb{P}^2 \) as a curve of degree 4. Thus

\[
\text{If } g = 3, \text{ X is isomorphic to a hyperelliptic curve or to a plane quartic.}
\]

If \( X \) is hyperelliptic, we can understand the canonical map. 
Represent \( X \) as the hyperelliptic curve constructed from 
The equation \( y^2 = h(x) \) where \( h \) has degree \( 2g+1 \) or \( 2g+2 \) and distinct roots.
By homework \( \frac{p(x) \, dx}{y} \) extends to a holomorphic 1-form on \( X \)
if \( p(x) \) is a polynomial of degree \( \leq g-1 \).

So \( \frac{dx}{y}, x \frac{dx}{y}, x^2 \frac{dx}{y}, \ldots, x^{g-1} \frac{dx}{y} \) is a basis of \( \Omega^1(X) \).

Let \( K = \text{div} \left( \frac{dx}{y} \right) \). Then \( L(K) = \langle 1, x, x^2, \ldots, x^{g-1} \rangle \).

In these coards, the canonical map \( \phi_k : X \to \mathbb{P}^{g-1} \)
\[ \phi_k(x, y) = [1 : x : x^2 : \cdots : x^{g-1}] \]

Thus \( \phi_k \) factors through the hyperelliptic projection \( \pi : X \to \mathbb{P}^1 \)
\( (x, y) \to [1 : x] \).

The map \( \mathbb{P}^1 \to \mathbb{P}^n \)
\[ [s : t] \to [s^n : s^{n-1}t : \cdots : t^n] \] is called a Veronese map.
(the image is the rational normal curve.)

Thus
\[ X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\nu} \mathbb{P}^{g-1} \]
\[ \phi_k = \nu \circ \pi \]
where \( \pi \) is hyperelliptic branched covering and \( \nu \) is the Veronese map.

On the other hand, suppose \( X_4 \subseteq \mathbb{P}^2 \) is defined by a nonsingular quartic equation: e.g. \( x^4 + y^4 + z^4 = 0 \).

Either by Hurwitz formula or by counting holomorphic 1-forms, we can see that the genus of \( X_4 \) is 3.
The linear system defining the embedding $X_4 \subseteq \mathbb{P}^2$ has degree 4 and dimension 3. Hence if $D$ is a hyperplane divisor for this embedding, we have deg $D = 4$ and $\ell(D) \geq 3$.

Now

$$\ell(D) = \ell(k-D) + 4 + 1 - 3 = \ell(k-D) + 2 \geq 3$$

So $\ell(k-D) \geq 1$.

But $\deg k = 2g - 2 = 2 \cdot 3 - 2 = 4$, so $\deg(k-D) = 0$.

Now $\deg(k-D) = 0$ and $\ell(k-D) \geq 1$ $\Rightarrow$ $k-D \sim 0 \Rightarrow k \sim D$.

Thus $D$ is a canonical divisor, $\ell(D) = \ell(k) = g = 3$, and so the embedding $X_4 \subseteq \mathbb{P}^2$ is exactly the canonical embedding of $X_4$.

Thus $X_4$ is not hyperelliptic.