Hard direction of Abel’s theorem, more properties of periods:

Recall notation from last lecture: $X$ compact Riemann surface
- $a_i, b_i, i = 1, \ldots, g$ basis of $H_1(X; \mathbb{Z})$ coming from identifying $\omega_1, \ldots, \omega_g$ basis of $\Omega^1(X)$
- $A_i(w) = \int_{a_i} \omega$, $B_i(w) = \int_{b_i} \omega$, periods
- $A = (A_i(w_j))_{i,j=1,\ldots,g}$, $B = (B_i(w_j))_{i,j=1,\ldots,g}$, period matrices.

We know $A$ and $B$ are nonsingular $g \times g$ matrices.

We proved the Riemann bilinear relations:

$$\sum_{i=1}^{g} \left[ A_i(w_1) B_i(w_2) - A_i(w_2) B_i(w_1) \right] = 0 \quad \text{(any $w_1, w_2 \in \Omega^1(X)$)}$$

$$\text{Im} \sum_{i=1}^{g} A_i(w) \overline{B_i(w)} < 0 \quad \text{(0 \neq w \in \Omega^1(X))}$$

The first relation is equivalent to $A^T B = B^T A$:

indeed, let $w_j, w_k$ be elements of our basis. The relation says

$$\sum_{i=1}^{g} A_i(w_j) B_i(w_k) = \sum_{i=1}^{g} B_i(w_j) A_i(w_k)$$

LHS = $i^j$ entry of $A^T B$, RHS = $j^k$ entry of $B^T A$.

Conversely, $A^T B = B^T A$ says relation holds for basis elements

which implies it for all $w_1, w_2 \in \Omega^1(X)$ by linearity

With this in hand, we can prove the hard direction of Abel’s theorem.
Lemma Suppose $D \in \text{Div}_0(X)$ and $A_0(D) = 0$ in $\text{Jac}(X)$

Then there is a meromorphic 1-form $\omega$ on $X$ such that

(i) $\omega$ has simple poles at points $p$ where $D(p) \neq 0$,
and is holomorphic elsewhere.

(ii) $\text{Res}_p(\omega) = D(p)$ for all $p \in X$

(iii) The $a$- and $b$-periods of $\omega$ are multiples of $2\pi i$

Proof By the existence of meromorphic differentials with prescribed
residues, there exists $\tau$ satisfying (i) and (ii):
the hypothesis $D \in \text{Div}_0(X)$ means $\sum_{P} D(P) = 0$.

Given one form $\tau$ satisfying (i) and (ii) any other is of the
form $\omega = \tau - \sum_{i=1}^{g} c_i \omega_i$, where $\omega_i$ is the basis of $\Omega^1(X)$.

The goal is to use the hypothesis $A_0(D) = 0$ to show that we
can pick constants $c_i$ so that $\omega$ satisfies (iii) also.
This involves some computations on the $4g$-gon $P$. We assume
no point where $D(p) \neq 0$ lies on $\partial P$; we may always achieve
this by wiggling the $a$ and $b$-curves.

Let $S_k = \frac{1}{2\pi i} \sum \left[ A_i(\omega_k) B_i(\tau) - A_i(\tau) B_i(\omega_k) \right]$

The integrals $A_i(\tau) = \int_{\alpha_i} \tau$, $B_i(\tau) = \int_{\beta_i} \tau$ make sense even though
$\tau$ is meromorphic, since $\tau$ has no poles along $\alpha_i$, $\beta_i$.

The formula $S_k = \frac{1}{2\pi i} \int_{\partial P} f_{\omega_k} \tau$ is valid with the same proof as before.

Thus $S_k = \frac{1}{2\pi i} \int_{\partial P} f_{\omega_k} \tau = \sum_{p \in P} \text{Res}_p(f_{\omega_k} \tau) = \sum_{p \in P} \text{Res}_p(f_{\omega_k} \tau)$
Where \( f_{wk} \) is a regarded as a form on \( X \) that is discontinuous across the \( a \)- and \( b \)-curves but which is otherwise holomorphic.

At a pole \( p \) of \( \tau \), \( f_{wk} \) is holomorphic and \( \tau \) has a simple pole, so
\[
\text{Res}_p (f_{wk} \tau) = f_{wk}(p) \text{Res}_p (\tau) = f_{wk}(p) D(p)
\]
Also \( f_{wk}(p) = \int_{p_0}^p \omega_k \) for a contour within \( p \), so
\[
s_k = \sum_p D(p) f_{wk}(p) = \sum_p D(p) \int_{p_0}^p \omega_k
\]

But this is precisely the \( k \)th component of \( A_0(D) \)!

i.e., \( A_0(D) = (s_1, s_2, ..., s_g)^T \)

The hypothesis \( A_0(D) = 0 \) in \( \text{Jac}(X) \) means \( (s_1, ..., s_g)^T \in \Delta \).

Thus \( (s_1, ..., s_g) \) is an integral linear combination of the rows of \( A \) and \( B \). So \( \exists \ m_i, n_i \in \mathbb{Z} \) such that
\[
s_k = \sum_{i=1}^g m_i A_i(w_k) - \sum_{i=1}^g n_i B_i(w_k) \quad (k=1, ..., g)
\]

But by definition \( s_k = \sum_{i=1}^g \frac{B_i(\tau)}{2\pi i} A_i(w_k) - \sum_{i=1}^g \frac{A_i(\tau)}{2\pi i} B_i(w_k) \)

meaning \( \sum_{i=1}^g (B_i(\tau) - 2\pi i m_i) A_i(w_k) = \sum_{i=1}^g (A_i(\tau) - 2\pi i n_i) B_i(w_k) \)

let \( b \) be column vector with \( b_i = B_i(\tau) - 2\pi i m_i \)

let \( a \) be column vector with \( a_i = A_i(\tau) - 2\pi i m_i \)
Then we have \( A^Tb = B^Ta \) is equivalent to the previous equation.

\textbf{Claim:} \exists c \text{ such that } Ac = a \text{ and } Bc = b

\textbf{Proof of claim:} Consider the sequence

\[ 0 \rightarrow \mathbb{C}^g \xrightarrow{\alpha = \begin{pmatrix} A \\ IB \end{pmatrix}} \mathbb{C}^q \xrightarrow{\beta = (B^T - A^T)} \mathbb{C}^q \xrightarrow{\gamma} \mathbb{C}^q \rightarrow 0 \]

Then \( \beta \alpha = B^TA - A^TIB = 0 \). Since \( A \) and \( IB \) are nonsingular, both \( \alpha \) and \( \beta \) have full rank \( q \). It follows that this sequence is exact. The vector \( \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^q \) satisfies

\[ \beta \begin{pmatrix} a \\ b \end{pmatrix} = B^Ta - A^Tb = 0 \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \ker(\beta) = \im(\alpha) \]

Thus \( \exists c \in \mathbb{C}^q \) such that \( Ac = \begin{pmatrix} a \\ b \end{pmatrix} \), i.e., \( Ac = a \) \( Bc = b \).

Now given such \( c = (c_j)_{j=1, \ldots, q} \), consider the term \( \omega = z - \sum c_j w_j \)

Then \( \omega \) satisfies (i) and (ii) and

\[ A_i(\omega) = A_i(t) - \sum_j c_j A_i(w_j) \]

\[ = A_i(t) - (AC)_i = A_i(t) - (a)_i \text{ since } Ac = a \]

\[ = A_i(t) - [A_i(t) - 2\pi i n_i] = 2\pi i n_i \]

Similarly,

\[ B_i(\omega) = 2\pi i n_i \text{ since } IBc = b \]

Thus \( \omega \) satisfies (iii) as well.

Now we can prove the hard direction of Abel's theorem.
Proposition: Let $D \in \text{Div}_0(X)$ and suppose $A_0(D) = 0$ in $\text{Jac}(X)$. Then $D = \text{div}(f)$ for some meromophic function $f$.

Proof: Use Lemma to construct $\omega$ satisfying (i), (ii), (iii).

Then set

$$f(z) = \exp \left( \int_{p_0}^{x} \omega \right) \quad (\text{any path } p_0 \to x)$$

To show $f$ is well defined, we need to show that if we change the path $p_0 \to x$, then $\int_{p_0}^{x} \omega$ changes by an integer multiple of $2\pi i$. Since the residues of $\omega$ are integers $D(p)$, moving the path across a pole $p$ changes $\int_{p_0}^{x} \omega$ by $2\pi i D(p)$.

Changing the homotopy class of the path in $X$ changes $\int_{p_0}^{x} \omega$ by a combination of $a$ and $b$-periods of $\omega$.

By (iii) these are integer multiples of $2\pi i$.

Thus $f(z)$ is well defined, at least away from the poles of $\omega$, and it is holomorphic there, and does not vanish there.

Let $p$ be a point where $D(p) \neq 0$, so $\omega$ has a simple pole of residue $D(p)$ there. In a local coord $z$ centered at $p$, we have

$$\omega = \frac{D(p)}{z} + g(z), \quad g(z) \text{ holomorphic near } p.$$ 

Therefore $\int_{p_0}^{x} \omega = D(p) \ln z + h(z)$ for $h(z)$ holomorphic near $p$.

So $f(z) = z^{D(p)} e^{h(z)}$ near $p$. Thus $f$ is meromorphic at $p$, and $\text{ord}_p(f) = D(p)$. Thus $\text{div}(f) = D$. 

\[ \square \]
More about Riemann bilinear relations.

Recall \( A, B \). They depend on choice of \( a \) and \( b \) curvves, and also on a choice of basis for \( \Omega^1(X) \).

Chern basis of \( \Omega^1(X) \):
\[
\omega'_j = \sum_k c_{kj} \omega_k
\]

\[
A_i(\omega'_j) = A_i(\sum_k c_{kj} \omega_k) = \sum_k c_{kj} A_i(\omega_k)
\]

i.e. \( A' = AC \) where \( C = (c_{kj}) \).

Setting \( C = A^{-1} \), we can achieve \( A' = I \) (identity matrix).

In short, by choosing an appropriate basis \( \omega'_j \) of \( \Omega^1(X) \), we may assume \( A = I \). Then \( IB \) is some other matrix, called the \underline{normalized period matrix}.

The relation \( A^TIB = B^TA \) becomes simply \( B = B^T \), i.e., \( B \) is symmetric.

As for \( \Im \sum_{i=1}^q A_i(\omega) \overline{B_i(\omega)} < 0 \) for all \( \omega \neq 0 \),

write \( \omega = \sum_{i=1}^q c_i \omega_i \) where \( \omega_i, \ldots, \omega_q \) is the basis of \( \Omega^1(X) \) that makes \( A = I \).

Then \( A_i(\omega) = c_i \) and \( B_i(\omega) = \sum_j c_j B_i(\omega_j) \)

So \( \Im \left( c_i \overline{c_j} B_i(\omega_j) \right) < 0 \) for any vector \( c = (c_i) \in \mathbb{C}^q \)

or \( 0 < \Im \left( \overline{c_i} c_j B_i(\omega_j) \right) = \Im (\overline{c^T} B c) \)

Taking \( c \) real, we see \( 0 < c^T(\Im B) c \) for any \( c \in \mathbb{R}^q \), \( c \neq 0 \).

This means \( \Im IB \) is positive definite.
The normalized period matrix is symmetric and has positive definite imaginary part.

**Corollary.** The 2g rows of $A$ and $B$ are linearly independent over $\mathbb{R}$, thus $\Lambda \subset \mathbb{C}^g$, which is the $\mathbb{Z}$-span of the transposes of these rows, is actually a lattice and $C^g/\Lambda \cong (\mathbb{R}/\mathbb{Z})^g$ as groups.

**Proof:** WLOG we may assume the period matrix is normalized so $A = I$ and $B = B^T$ with $\text{Im} \ B > 0$.

Suppose $C = \begin{pmatrix} a \\ b \end{pmatrix}$ is a vector $a, b \in \mathbb{R}^g$ such that

$$c^T \begin{pmatrix} I \\ B \end{pmatrix} = 0 \quad \text{i.e.} \quad (a^T \ b^T) \begin{pmatrix} I \\ B \end{pmatrix} = a^T + b^T B = 0$$

Multiply by $b$ on right: $a^T b + b^T B b = 0$

Take $\text{Im}$: $\text{Im} \ b^T B b = b^T (\text{Im} B) b = 0$

$\therefore \ b = 0$ since $\text{Im} \ B$ is positive definite

$\therefore \ a = 0$. Thus the rows of $\begin{pmatrix} I \\ B \end{pmatrix}$ are linearly independent over $\mathbb{R}$. \[\square\]