Abel's theorem in genus 1

Let \( X \) be a compact Riemann surface of genus 1. We pick an identification of \( X \) with polygon

\[
\begin{array}{c}
\text{\( a' \)} \\
\text{\( a \)} \\
\text{\( b' \)} \\
\text{\( b \)}
\end{array}
\]

\[
\text{Jac}(X) = \frac{\omega^1(X)}{H_1(X, \mathbb{Z})}
\]

As we know, \( \omega^1(X) \cong \mathbb{C} \), and \( K \sim 0 \), so \( X \) has a nowhere vanishing holomorphic differential \( \omega \), unique up to scalar multiple. We may normalize \( \omega \) so that

\[
\int_a \omega = 1.
\]

With this choice, set \( \tau = \int_b \omega \).

The period matrix \( \Pi B \) is \( 1 \times 1 : \Pi B = (\tau) \). The Riemann bilinear relations reduce to \( \text{Im} \tau > 0 \).

Thus \( \text{Jac}(X) \cong \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) for some \( \tau \in \mathbb{H} = \{ \text{Im} \tau > 0 \} \) upper-half plane.

Choose a base point \( p_0 \in X \), and let

\[
A_1: X \to \text{Jac}(X)
\]

be the Abel-Jacobi map

\[
A_1(p) = \int_{p_0}^p \omega \in \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})
\]

This map is injective, for if \( A_1(p) = A_1(q) \) then \( A_0(p-q) = 0 \), where \( A_0: \text{Div}(X) \to X \) is the map in degree two divisors.

By (the hard direction of) Abel's theorem, this means

\( p-q = \text{div}(f) \) for some \( f \in \mathcal{M}(X) \). If \( p \neq q \),

this \( f \) would have a single simple pole, which would give a degree \( 1 \) map to \( \mathbb{C} \), so \( X \cong \mathbb{C} \), which is absurd.
Also, \( A_1 : X \to \text{Jac}(X) \) is holomorphic; indeed
\[
\frac{d}{dp} A_1(p) = \frac{d}{dp} \int_{p_0}^{p} \omega = \omega \quad \text{which is holomorphic.}
\]

Thus \( A_1 \) is an injective holomorphic map between compact Riemann surfaces, hence has degree 1, hence is an isomorphism.

Theorem: Every genus 1 Riemann surface is isomorphic to a complex torus \( \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}) \) for some \( t \in \mathbb{H} \).

It is interesting to study the inverse isomorphism
\[
\Phi : \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}) \to X
\]

For example, suppose \( X \) is a smooth cubic curve in \( \mathbb{P}^2 \).

Then \( X \) has genus 1, and we get an isomorphism
\[
\Phi : \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}) \to X.
\]

Let \( f \) be a meromorphic function on \( X \), such as \( \frac{x}{z} \) or \( \frac{y}{z} \), where \([x:y:z]\) are homogeneous coordinates on \( \mathbb{P}^2 \).

Then \( \Phi^*f \) is a meromorphic function on \( \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}) \), or in other terms a \((\mathbb{Z} + t\mathbb{Z})\)-periodic
meromorphic function on \( \mathbb{C} \). Such a function is called an Elliptic Function. (look up Weierstrass \( \wp \)).

The relationship between \( t \) and the coefficients of the defining equation of \( X \) is also interesting.

The curve \( y^2 = 4x^3 - g_2x - g_3 \) has normalized period \( \tau \) where
\[
g_2 = 60 \sum_{n,m \in \mathbb{Z} \not \text{ both zero}} (n + m\tau)^{-4} \quad g_3 = 140 \sum_{n,m \in \mathbb{Z} \not \text{ both zero}} (n + m\tau)^{-6}
\]
These \( g_2(t) \) and \( g_3(t) \) are examples of modular forms. There is much more to say about elliptic functions and modular forms, but let us turn to the question “How many Riemann surfaces of genus \( g \) are there, up to isomorphism?” The answer is 1 if \( g = 0 \) and continuum-many if \( g > 0 \). In the latter case, we seek to parametrize all Riemann surfaces of genus \( g \) by points in a space:

\[
M_g = \text{moduli space of genus } g \text{ Riemann surfaces.}
\]

At the moment, \( M_g \) is nothing but a name, but whatever it is, each Riemann surface \( X \) should give a point \([X]\) such that \([X] = [X'] \iff X \cong X'\), i.e. points of \( M_g \) correspond bijectively to isomorphism classes of Riemann surfaces.

Riemann realized that \( M_g \) should be a complex “manifold” of complex dimension \( 3g - 3 \). The textbook has an account of his arguments.

One approach to \( M_g \) goes through the period theory. This is an association

\[
(X, (a_1, \ldots, a_g, b_1, \ldots, b_g)) \mapsto \begin{pmatrix} \mathbf{B} \end{pmatrix} \in M_{g \times g}(\mathbb{C}) \text{ s.t. } \mathbf{B}^T = \mathbf{B} \text{ and } \text{Im } \mathbf{B} > 0
\]

Where \( \mathbf{B} = (b_{ij}) \) where \( b_{ij} = \delta_{ij} \).

This is the “normalized period matrix”, it depends on the choice of \((a_1, \ldots, a_g, b_1, \ldots, b_g)\) coming from an identification with the \( 4g \)-gon with sides glued.
If we change the choice of identification, we get another basis \( a'_1, \ldots, a'_g, b'_1, \ldots, b'_g \) of \( H_1(X;\mathbb{Z}) \).

Since \( H_1(X;\mathbb{Z}) \cong \mathbb{Z}^{2g} \), any two bases are related by some matrix \( \Gamma \in \text{GL}(2g,\mathbb{Z}) \). But there is an aspect we are missing. The group \( H_1(X;\mathbb{Z}) \) carries a natural skew-symmetric bilinear form, the intersection pairing,

\[
E : H_1(X;\mathbb{Z}) \otimes H_1(X;\mathbb{Z}) \to \mathbb{Z}
\]

\[(\alpha, \beta) \mapsto \alpha \cdot \beta\]

where \( \alpha \cdot \beta \) is the signed count of intersections of the oriented loops \( \alpha \) and \( \beta \)

\[
\begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\end{array}
\xrightarrow{\text{oriented}} +1 \\
\begin{array}{c}
\beta \\
\uparrow \\
\alpha \\
\end{array}
\xrightarrow{\text{oriented}} -1
\]

where we use the complex orientation of \( X \) to map a neighborhood of the intersection point onto the plane.

We have \( a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \)

where \( (a_i, b_i) \) is any basis coming from the identification with the polygon.

If \( (a_i, b_i) \) and \( (a'_i, b'_i) \) are two such bases, then the change of basis matrix \( \Gamma \in \text{GL}(2g,\mathbb{Z}) \) actually lies in the subgroup

\[
\text{Sp}(2g,\mathbb{Z}) = \left\{ \Gamma \in \text{GL}(2g,\mathbb{Z}) \mid \Gamma^T J \Gamma = J \right\}
\]

where \( J = \begin{pmatrix} 0 & I_{g \times g} \\ -I_{g \times g} & 0 \end{pmatrix} \) represents intersection form.

**Symplectic group.**
Let \( \mathcal{H}_q = \{ P \in \text{M}_{g \times g}(C) \mid P^T = P \text{ and } \text{Im } P > 0 \} \)

(The Siegel upper-half-space: think \( P = \mathcal{B} \) from above)

There is an action of \( \text{Sp}(2g, \mathbb{Z}) \) on \( \mathcal{H}_q \): write \( \Gamma \in \text{Sp}(2g, \mathbb{Z}) \) as \( \Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) in \( g \times g \) block form, and set

\[
\Gamma \cdot P = (\Gamma C + D)^{-1}(\Gamma A + B)
\]

This is a group action on \( \mathcal{H}_q \), and the quotient is denoted \( \mathcal{A}_q = \mathcal{H}_q / \text{Sp}(2g, \mathbb{Z}) \)

To each compact Riemann surface \( X \) of genus \( g \), there is a well-defined point \( [\mathcal{B}] \in \mathcal{H}_q / \text{Sp}(2g, \mathbb{Z}) = \mathcal{A}_q \) given by the orbit of the normalized period matrix.

This amounts to a map \( \mathcal{M}_g \to \mathcal{A}_q \).

The Torelli theorem says this map is injective:

**Theorem.** If \( X_1 \) and \( X_2 \) are genus \( g \) compact R.S., and they have normalized period matrices \( \mathcal{B}_1, \mathcal{B}_2 \) that represent the same point in \( \mathcal{A}_q \), then \( X_1 \cong X_2 \).

Now \( \mathcal{A}_q \) is a complex "manifold" of dimension \( \frac{g(g+1)}{2} \)

So we can regard \( \mathcal{M}_g \) as a "submanifold" of \( \mathcal{A}_q \), of dimension \( 3g-3 \). Surprisingly, exactly what submanifold it is is the Schottky problem. There are solutions available, but none are very explicit.
For $g=1$, the construction of $\mathbb{C}/\mathbb{Z}+\mathbb{Z}$ shows $M_1 = \mathbb{A}_1$.

For $g=2$ or $3$, $\dim M_g = \dim A_g$, and $M_g$ is an open subset of $A_g$.

For $g=4$, $\dim M_4 = 9$, $\dim A_4 = 10$, and there is a classical description due to Schottky.

For $g \geq 5$, not explicitly known what $M_g \setminus A_g$ looks like.

Due to this difficulty, other constructions of $M_g$ are usually preferred, based on Teichmüller theory or geometric invariant theory.

$q=1$ case: $M_1 = \mathbb{H}_1 = \mathbb{H}/\text{Sp}(2,\mathbb{Z})$

$\mathbb{H}_1 = \mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im} \tau > 0 \}$ upper half-plane

$\text{Sp}(2,\mathbb{Z}) = \text{SL}(2,\mathbb{Z}) = \{ (a \ b) \ c \ d \mid ad-bc = 1, a,b,c,d \in \mathbb{Z} \}$

$(a \ b) \cdot \tau = \frac{a\tau + b}{c\tau + d}$ is the action. (Möbius transformation)

Thus the moduli space of genus 1 Riemann surfaces is $\mathbb{H}/\text{SL}(2,\mathbb{Z})$. 