Math 510 textbook supplement: Serre’s proof of Riemann-Roch written in Miranda’s notation

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November 7, 2019

In R. Miranda’s textbook *Algebraic Curves and Riemann Surfaces*, the proof of the Riemann-Roch theorem occupies chapter VI. This proof appears to be based on the one presented by J.-P. Serre in *Groupes algébriques et corps de classes*, chapitre II. The main difference is that the notion of répartition used by Serre and otherwise known to number theorists under the name *adèle* is replaced by that of *Laurent tail divisor*, which is unknown outside of Miranda’s text. Laurent tail divisors are certain non-canonical representatives in quotient spaces of the adéle ring.

The difference may seem inessential, but as a result of this less natural choice, Miranda is forced to formulate and prove a rather baffling lemma (Lemma VI.3.4). This lemma comes at the most difficult part of the proof, namely the “representability” aspect of Serre duality (that every linear functional on $H^1(D)$ is represented by residue pairing with a differential). This seems to correspond to Serre’s Proposition 4, which is no less miraculous, but whose statement seems clearer.

Unfortunately, Serre’s book and Miranda’s are not completely compatible. For instance, Serre assumes familiarity with coherent sheaves, which Miranda does not, but Serre does not assume the residue theorem, which Miranda has already proved by this point. Also the notation is slightly different. This note will follow Miranda’s notation as closely as possible, so that it can be read along with sections VI.2 and VI.3 of that book.

1 Setup

Let $X$ be a compact Riemann surface. Since everything below depends on $X$, we will omit it from the notation everywhere. Let $D$ be a divisor.

- $\mathcal{M}$ is the field of meromorphic functions on $X$.
- $\mathcal{M}^{(1)}$ is the space of meromorphic differentials on $X$. It is a one-dimensional $\mathcal{M}$-vector space.
- $L(D)$ is the space of meromorphic functions $f$ such that $\text{div}(f) + D \geq 0$. 
- $L^{(1)}(D)$ is the space of meromorphic differentials $\omega$ such that $\text{div}(\omega) + D \geq 0$.
- Suppose $D_1 \leq D_2$. Then $L(D_1) \subseteq L(D_2)$ and $L^{(1)}(D_1) \subseteq L^{(1)}(D_2)$. Furthermore, $\mathcal{M} = \bigcup_D L(D)$ and $\mathcal{M}^{(1)} = \bigcup_D L^{(1)}(D)$.
- For $f \in \mathcal{M}^\times$, multiplication by $f$ maps $L(D)$ isomorphically onto $L(D - \text{div}(f))$, and maps $L^{(1)}(D)$ isomorphically onto $L^{(1)}(D - \text{div}(f))$.
- If $f \in L(D_1)$, $g \in L(D_2)$, and $\omega \in L^{(1)}(D_3)$, then $fg \in L(D_1 + D_2)$ and $f\omega \in L^{(1)}(D_1 + D_3)$.
Proposition 1. The spaces $L(D)$ and $L^{(1)}(D)$ are finite dimensional over $\mathbb{C}$.

Proof. This is Proposition 3.16.

2 Répartitions

A répartition is a family $(r_P)_{P \in X}$ of meromorphic functions $r_P \in \mathcal{M}$ indexed by points $P \in X$, with the property that for almost all $P$, $\text{ord}_P(r_P) \geq 0$. We denote $R$ the set of répartitions:

$$R = \{(r_P)_{P \in X} \mid r_P \in \mathcal{M}, \text{ord}_P(r_P) \geq 0 \text{ for all but finitely many } P \in X\}$$

This set is a ring under componentwise operations. There is a homomorphism of rings $\alpha : \mathcal{M} \to R$ that sends $f \in \mathcal{M}$ to the répartition $(r_P)_{P \in X}$ such that $r_P = f$ for all $P \in X$. Note that for a general répartition there is no necessary relationship between the different components $r_P$ as $P \in X$ varies.

The space $R(D)$ is defined analogously to $L(D)$:

$$R(D) = \{(r_P) \in R \mid (\forall P \in X)(\text{ord}_P(r_P) + D(P) \geq 0)\}$$

The quotient space $R/R(D)$ will play a central rôle in what follows. (It is isomorphic to the space of Laurent tail divisors $T[D](X)$ defined by Miranda: see Section 6.) By composing $\alpha$ with the quotient map, we obtain a linear map

$$\alpha_D : \mathcal{M} \to R/R(D)$$

The kernel of $\alpha_D$ is nothing but $L(D)$. The cokernel of $\alpha_D$ is a new space, which we denote $H^1(D)$:

$$H^1(D) = R/(R(D) + \alpha(\mathcal{M}))$$

(This notation is compatible with Miranda; Serre denotes this space by $I(D)$.)

Observe that if $D_1 \leq D_2$, then $R(D_1) \subseteq R(D_2)$, so there is a surjective map $R/R(D_1) \to R/R(D_2)$, and there is also a surjective map $H^1(D_1) \to H^1(D_2)$. The kernel of this last map is denoted $H^1(D_1/D_2)$.

Proposition 2. The space $H^1(D)$ is finite dimensional over $\mathbb{C}$.

Proof. The arguments on pp. 181–184 of Miranda's text establish this if we read $R/R(D)$ in place of $T[D](X)$.

Theorem 3 (Riemann-Roch theorem: first form). For any divisor $D$,

$$\dim L(D) - \dim H^1(D) = \deg(D) + 1 - \dim H^1(0).$$

Proof. Again, the argument from Miranda works just as well.

3 Dual spaces to répartition quotients

Consider the space of all $\mathbb{C}$-linear functionals on $R$, and the subspace thereof consisting of functionals that vanish on $\alpha(\mathcal{M})$ and $R(D)$ for some $D$:

$$J = \{\lambda : R \to \mathbb{C} \mid (\exists D)(\lambda|_{R(D) + \alpha(\mathcal{M})} = 0)\}$$
A slight variation is to consider the subspace of functionals that vanish on \( R(D) + \alpha(\mathcal{M}) \) for a given \( D \); this space is denoted \( J(-D) \) or in other words:

\[
J(D) = \{ \lambda : R \to \mathbb{C} \mid \lambda |_{R(-D)+\alpha(\mathcal{M})} = 0 \}.
\]

The minus sign is there to ensure that if \( D_1 \preceq D_2 \), then \( J(D_1) \subseteq J(D_2) \). Also we have \( J = \bigcup_D J(D) \).

Next observe that, via the homomorphism \( \alpha : \mathcal{M} \to R \), the ring \( R \) has the structure of an \( \mathcal{M} \)-vector space. Hence its dual space is also an \( \mathcal{M} \)-vector space, via the action \( (f, \lambda)(r) = \lambda(\alpha(f)r) \). The subspace \( J \) is stable under this action, and so becomes an \( \mathcal{M} \)-vector space.

- \( J(-D) \) is naturally identified with the \( \mathbb{C} \)-linear dual of \( H^1(D) \):
  \[
  J(-D) \cong H^1(D)^*.
  \]

- Multiplication by \( f \in \mathcal{M}^\times \) maps \( J(D) \) isomorphically onto \( J(D - \text{div}(f)) \).

- If \( f \in L(D_1) \) and \( \lambda \in J(D_2) \), then \( f \lambda \in J(D_1 + D_2) \).

**Proposition 4.** The dimension of \( J \) as an \( \mathcal{M} \)-vector space is at most one.

In fact, the dimension is exactly one, but this will follow later.

**Proof.** Suppose for a contradiction that \( \alpha \) and \( \beta \) are two elements of \( J \) that are linearly independent over \( \mathcal{M} \). Then the function \( \mathcal{M} \oplus \mathcal{M} \to J \) given by \( (f, g) \mapsto f \alpha + g \beta \) is injective. Since \( J = \bigcup_D J(D) \), there is some divisor \( D \) such that \( \alpha, \beta \in J(D) \). Now choose a divisor \( \Delta_n \) of degree \( n \), for instance we could take \( nP \) where \( P \in X \) is some point. For any \( f, g \in L(\Delta_n) \), we have \( f \alpha + g \beta \in J(D + \Delta_n) \). Thus by linear independence there is an injective function \( L(\Delta_n) \oplus L(\Delta_n) \to J(D + \Delta_n) \), which implies that for every \( n \),

\[
2 \dim L(\Delta_n) \leq \dim J(D + \Delta_n) = \dim H^1(-D - \Delta_n).
\]

To obtain a contradiction, we show that the inequality above is not compatible with the first form of the Riemann-Roch theorem as \( n \to \infty \). We have

\[
\dim H^1(-D - \Delta_n) = \dim L(-D - \Delta_n) - \deg(-D - \Delta_n) - 1 + \dim H^1(0)
= \dim L(-D - \Delta_n) + \deg(D) + n - 1 + \dim H^1(0).
\]

Now if \( n > -\deg(D) \), then \( \deg(-D - \Delta_n) < 0 \), so \( L(-D - \Delta_n) = 0 \). This means that for large \( n \),

\[
\dim H^1(-D - \Delta_n) = n + C
\]

where \( C = \deg(D) - 1 + \dim H^1(0) \) is a constant.

On the other hand, we have that \( \dim L(\Delta_n) \geq n + 1 - \dim H^1(0) \), so \( 2 \dim L(\Delta_n) \geq 2n + C' \), where \( C' = 2(1 - \dim H^1(0)) \) is a constant. Combining this with the inequality above leads to

\[
2n + C' \leq n + C
\]

for all large \( n \), which is absurd. \( \square \)

We will see that the preceding proposition, though somewhat strange, contains the most difficult part of the Serre duality theorem.
4 Residues and duality

Take \( \omega \in \mathcal{M}^{(1)} \) and \( r \in R \). We can form their residue pairing

\[
\langle \omega, r \rangle = \sum_{P \in X} \text{Res}_P (\omega r_P).
\]

This is actually a finite sum, because for all but finitely many \( P \), both \( \omega \) and \( r_P \) are holomorphic at \( P \). The pairing is bilinear, and so it defines a map \( \beta : \mathcal{M}^{(1)} \to R^* \), \( \omega \mapsto \langle \omega, - \rangle \).

- If \( \omega \in \mathcal{M}^{(1)} \) and \( f \in \mathcal{M} \), then
  \[
  \langle \omega, \alpha(f) \rangle = \sum_{P \in X} \text{Res}_P (\omega f) = 0,
  \]
  where we have used the residue theorem and the fact that \( \omega f \) is a global meromorphic differential. Thus \( \beta(\omega) \) vanishes on \( \alpha(\mathcal{M}) \).

- If \( \omega \in L^{(1)}(-D) \), and \( r \in R(D) \), then for every \( P \in X \),
  \[
  \text{ord}_P (\omega r_P) = \text{ord}_P (\omega) + \text{ord}_P (r_P) \geq D(P) - D(P) = 0
  \]
  and so \( \langle \omega, r \rangle = 0 \). Thus, if \( \omega \in L^{(1)}(-D) \) then \( \beta(\omega) \) vanishes on \( R(D) \).

These two observations imply that if \( \omega \in L^{(1)}(-D) \), then \( \beta(\omega) = \langle \omega, - \rangle \in J(-D) \). In other words, for every \( D \), the residue pairing defines a \( \mathbb{C} \)-linear map

\[
\beta_D : L^{(1)}(D) \to J(D) \cong H^1(-D)^*.
\]

Taking the union over all \( D \), we obtain a map

\[
\beta : \mathcal{M}^{(1)} \to J,
\]

and we can see that this map is \( \mathcal{M} \)-linear.

**Theorem 5** (Serre duality). The maps \( \beta : \mathcal{M}^{(1)} \to J \) and \( \beta_D : L^{(1)}(D) \to J(D) \) are isomorphisms.

**Proof.** We first show that \( \beta \) is an isomorphism. Since \( \mathcal{M}^{(1)} \) is an \( \mathcal{M} \)-vector space of dimension one, and \( J \) is a \( \mathcal{M} \)-vector space of dimension at most one, it suffices to show that \( \beta \) is not the zero map (which also establishes that \( J \) is not the zero vector space and so has \( \mathcal{M} \)-dimension exactly one).

Choose a nonzero \( \omega \in \mathcal{M}^{(1)} \), and pick a point \( P_0 \in X \). Let \( k = -1 - \text{ord}_{P_0} (\omega) \). Let \( g \) be a meromorphic function on \( X \) such that \( \text{ord}_{P_0} (g) = k \) (which exists since \( X \) is an algebraic curve in Miranda's sense). Now construct a répartition \( r \) by setting \( r_{P_0} = g \) and \( r_P = 0 \) for all \( P \neq P_0 \). Then

\[
\beta(\omega)(r) = \langle \omega, r \rangle = \text{Res}_{P_0} (\omega g)
\]

Now \( \text{ord}_{P_0} (\omega g) = \text{ord}_{P_0} (\omega) + k = -1 \), and a simple pole must have a nonzero residue, so \( \beta(\omega) \neq 0 \).

A refinement of the argument just given shows that \( \beta_D \) is also an isomorphism. Injectivity of \( \beta \) automatically implies injectivity of \( \beta_D \). Surjectivity of \( \beta \) implies that every \( \lambda \in J(D) \) is of the form \( \lambda = \beta(\omega) \) for some \( \omega \); we just need to show that such \( \omega \) is actually in \( L^{(1)}(D) \).

Thus we claim: if \( \beta(\omega) \in J(D) \), then \( \omega \in L^{(1)}(D) \). Suppose not. Then there is a point \( P_0 \) such that \( \text{ord}_{P_0} (\omega) + D(P_0) < 0 \). As before, construct the répartition \( r \) which has order \( k = -1 - \text{ord}_{P_0} (\omega) \) at \( P_0 \) and which is zero at all other points. Then \( \beta(\omega)(r) \neq 0 \). On the other hand, \( \text{ord}_{P_0} (\omega) + D(P_0) < 0 \) means \( -\text{ord}_{P_0} (\omega) - D(P_0) > 0 \), so \( -1 - \text{ord}_{P_0} (\omega) - D(P_0) \geq 0 \). But this means that \( r \in R(-D) \), and since \( \beta(\omega) \) was supposed to be in \( J(D) \), we must have \( \beta(\omega)(r) = 0 \). Contradiction.

**Corollary 6.** \( \dim H^1(D) = \dim J(-D) = \dim L^{(1)}(-D) = \dim L(K - D) \), where \( K \) is a canonical divisor.
5 The Riemann-Roch theorem

Combining the first form of the Riemann-Roch theorem with Serre duality, we obtain
\[
\dim L(D) - \dim L(K - D) = \deg D + 1 - \dim L(K)
\]

Notice that \( L(K) \cong L(1)(0) \) is isomorphic to the space of holomorphic differentials on \( X \). We wish to compute the number \( \dim L(K) \). Apply the Riemann-Roch theorem with \( D = K \) to obtain
\[
\dim L(K) = \deg K + 1 - \dim L(0)
\]
\[
2 \dim L(K) = \deg K + 2
\]

Now we have already shown using the Riemann-Hurwitz formula that \( \deg K = 2g - 2 \), which combined with this equation implies \( \dim L(K) = g \), where \( g \) is the topological genus of \( X \).

This is the last bit of information needed to prove the final form of the Riemann-Roch theorem.

**Theorem 7** (Riemann-Roch theorem: final form). *Let \( D \) be a divisor on a compact Riemann surface \( X \) of genus \( g \), and let \( K \) be a canonical divisor. Then*
\[
\dim L(D) - \dim L(K - D) = \deg D + 1 - g.
\]

6 Appendix: Comparing \( R/R(D) \) and \( T[D](X) \)

In this section we construct an isomorphism between \( R/R(D) \) and \( T[D](X) \), the space of so-called Laurent tail divisors used by Miranda. First recall the definition: For each \( P \in X \), choose once and for all a local coordinate \( z_p \) centered at \( P \). A *Laurent tail divisor* is a finite formal sum
\[
\sum_P s_P(z_p) \cdot P
\]
where each coefficient \( s_P(z_p) \) is a Laurent polynomial in \( z_p \). Put differently, the abelian group of Laurent tail divisors is direct sum of the groups of Laurent polynomials in each of the coordinates \( z_p \).

Observe that a non-zero Laurent polynomial \( f(z) \) has a *maximum degree*, \( \maxdeg(f) \), which is the largest exponent appearing in \( f(z) \). (It also has a *minimum degree*, \( \mindeg(f) \), but this is merely an alias for \( \ord_{z=0}(f) \).) For each divisor \( D \), we define
\[
T[D](X) = \left\{ \sum_P s_P(z_p) \cdot P \mid \maxdeg(s_P) < -D(P) \right\}.
\]

Next we define a map \( \psi_D : R \to T[D](X) \). Given \( r = (r_P)_{P \in X} \), we let \( \psi_D(r) = \sum_P s_P(z_p) \cdot P \), where \( s_P(z_p) \) is obtained from \( r_P \in \mathcal{M} \) by taking the Laurent series expansion of \( r_P \) in the coordinate \( z_p \), and then deleting all terms with degrees \( \geq -D(P) \). This prescription does indeed result in a finite sum: for any given \( r \), it is true that outside of a finite set of points, we have both \( D(P) = 0 \) and \( \ord_P(r_P) \geq 0 \), so \( s_P(z_p) = 0 \).

**Proposition 8.** The map \( \psi_D : R \to T[D](X) \) is a surjective homomorphism of abelian groups whose kernel is \( R(D) \). Therefore it descends to an isomorphism \( \overline{\psi}_D : R/R(D) \to T[D](X) \).

**Proof.** To see that \( \psi_D \) is surjective, let \( s = \sum_P s_P(z_p) \cdot P \in T[D](X) \). For each \( P \) in the support of \( s \), choose a meromorphic function \( r_P \) whose tail at \( P \) is \( s_P(z_p) \). For all \( P \) not in the support of \( s \), let \( r_P = 0 \). Then \( r = (r_P)_{P \in X} \) and \( \psi_D(r) = s \).

The kernel of \( \psi_D \) consists of those répartitions \( r = (r_P)_{P \in X} \) such that, for all \( P \in X \), the Laurent expansion of \( r_P \) in the coordinate \( z_p \) has no terms of degree \( < -D(P) \). This is just to say that \( \ord_P(r_P) \geq -D(P) \) for all \( P \), and this is the defining condition of \( R(D) \).